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Probability and Analysis
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The Schrödinger equation for the fractional Laplacian in negative curvature

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Equations

Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) - \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases}$$

Half wave equation

$$\begin{cases} i\,\partial_t u(t,x) + \sqrt{-\Delta_x}\,u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases}$$

Schrödinger equation for the fractional Laplacian

$$\begin{cases} i\,\partial_t u(t,x) + (-\Delta_x)^{\frac{\varkappa}{2}} u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \tag{1}$$

where $1 < \varkappa < 2$ (possibly also $0 < \varkappa < 1$)

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Standard strategy

- Linear homogeneous equation : $F = 0$
  - Solution :

$$u(t,x) = e^{\,i\,t\,(-\Delta)^{\varkappa/2}} f(x) = \int k_t^{\varkappa}(x,y)\, f(y)\, dy$$

  - **Kernel estimate**
  - **Dispersive estimate :**

$$\big\| e^{\,i\,t\,(-\Delta)^{\varkappa/2}} \big\|_{L^{q'} \to L^{q}} \qquad \forall\, t \in \mathbb{R}^*, \forall\, 2 \le q \le \infty$$

- Linear inhomogeneous equation : $F \neq 0$
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$$u(t,x) = e^{\,i\,t\,(-\Delta)^{\varkappa/2}} f(x) + \int_0^t e^{\,i(t-s)(-\Delta_x)^{\varkappa/2}} F(s,x)\, ds$$

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Standard strategy (continued)

- Nonlinear equation : $F(t,x)=\widetilde{F}(u(t,x))$
  with $\widetilde{F}(u)$ power–like

  e.g. $\widetilde{F}(u)=\text{const.}\begin{cases} u^\gamma & (\gamma \text{ integer} \geq 2) \\ |u|^\gamma & (\gamma>1) \\ u\,|u|^{\gamma-1} & (\gamma>1) \end{cases}$

  Main problem = local/global well–posedness
  $\sim$ existence and uniqueness of solutions

  Tools :
  - Strichartz inequality
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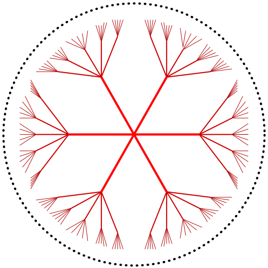
Homogeneous trees

$\mathbb{T} = \mathbb{T}_Q$ homogeneous tree with $Q+1 \geq 3$ edges

Example : $Q=5$



Discrete analogs of hyperbolic spaces

Volume of balls of radius $r \in \mathbb{N}$: $V(r) = 1 + \frac{Q+1}{Q-1}(Q^r - 1) \asymp Q^r$

Combinatorial Laplacian on (the vertices of) $\mathbb{T}$:

$$\Delta f(x) = \frac{1}{Q+1} \sum_{d(y,x)=1} f(y) - f(x)$$

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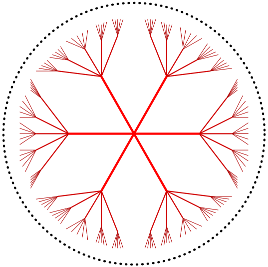

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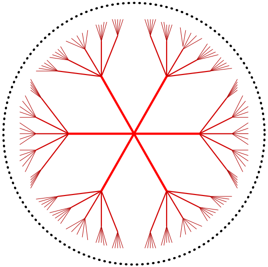

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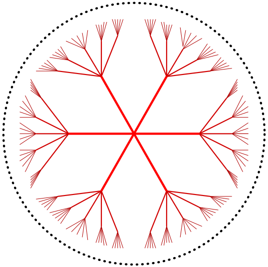

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Schrödinger equation on $\mathbb{T}$

On $\mathbb{T}$, the Schrödinger equation (with continuous time)

$$\begin{cases} i\,\partial_t u(t,x) + (-\Delta_x)^{\frac{\varkappa}{2}} u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \tag{1}$$

can be solved by using the Fourier transform :

$$u(t,x) = \underbrace{e^{\,it(-\Delta)^{\varkappa/2}} f(x)}_{\text{homogeneous}} + \underbrace{\int_0^t e^{\,i(t-s)(-\Delta_x)^{\varkappa/2}} F(s,x)\,ds}_{\text{inhomogeneous}}$$

where

$$e^{\,it(-\Delta)^{\varkappa/2}} f(x) = \underbrace{\sum\nolimits_{y\in\mathbb{T}} f(y)\,k_t^{\varkappa}(d(x,y))}_{f\,*\,k_t^{\varkappa}(x)}$$

Schrödinger kernel on $\mathbb{T}$

Inverse spherical Fourier transform

$$k_t^{\varkappa}(r) = \text{const.} \int_0^{\frac{\pi}{\log Q}} e^{\,it\,[1-\gamma(\lambda)]^{\varkappa/2}}\,\varphi_\lambda(r)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda$$

where

$$\gamma(\lambda) = \frac{Q^{i\lambda} + Q^{-i\lambda}}{Q^{1/2} + Q^{-1/2}}$$

$$\mathbf{c}(\lambda) = \frac{1}{Q^{1/2} + Q^{-1/2}}\,\frac{Q^{1/2+i\lambda} - Q^{-1/2-i\lambda}}{Q^{i\lambda} - Q^{-i\lambda}}$$

$$\varphi_\lambda(r) = \mathbf{c}(\lambda)\,Q^{(-1/2+i\lambda)r} + \mathbf{c}(-\lambda)\,Q^{(-1/2-i\lambda)r}$$

Kernel estimate on $\mathbb{T}$

Hence

$$k_t^{\varkappa}(r) = \text{const.}\, Q^{-r/2} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{\,it\,[1-\gamma(0)\cos\lambda]^{\varkappa/2} - i\,r\,\lambda} \times \frac{\sin\lambda}{Q^{1/2}\,e^{i\lambda} - Q^{-1/2}\,e^{-i\lambda}}\,d\lambda$$

By stationary phase analysis, one gets

Global upper bound

Assume that $0<\varkappa\le 2$. Then

$$|k_t^{\varkappa}(r)| \lesssim Q^{-\frac{r}{2}} \qquad \forall\, t\in\mathbb{R}^*,\ \forall\, r\in\mathbb{N}$$

Moreover there exists a constant $C>0$ such that

$$|k_t^{\varkappa}(r)| \lesssim |t|^{-\frac{3}{2}}\,(1+r)\,Q^{-\frac{r}{2}} \qquad \forall\, t\in\mathbb{R}^*,\ \forall\, r\in\mathbb{N}$$

if $1+r\le C\,|t|$

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Linear estimates on $\mathbb{T}$

Dispersive estimate

Let $0 < \varkappa \leq 2$ and $2 < q \leq \infty$. Then

$$\| e^{it(-\Delta)^{\varkappa/2}} \|_{\ell^{q'} \to \ell^q} \lesssim (1+|t|)^{-\frac{3}{2}} \qquad \forall\, t \in \mathbb{R}^*$$

Main tool = following version of the Kunze–Stein phenomenon

Lemma

Let $2 \leq q < \infty$ and $\frac{q}{2} \leq p < q$. Then

$$L^{q'}(\mathbb{T}) * L^p_{\text{rad}}(\mathbb{T}) \subset L^q(\mathbb{T})$$

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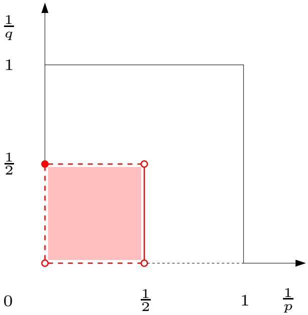
for all admissible pairs $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ in the following square



NLS on $\mathbb{T}$

Consider the nonlinear Schrödinger equation

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where

$$\begin{cases} |\widetilde{F}(u)| \lesssim |u|^{\gamma} \\ |\widetilde{F}(u) - \widetilde{F}(v)| \lesssim \{|u|^{\gamma-1} + |v|^{\gamma-1}\}\,|u-v| \end{cases}$$

for some exponent $\gamma > 1$

Theorem

- (2) is locally well–posed for arbitrary initial data in ℓ^2
- (2) is globally well–posed for small initial data in ℓ^2
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Hyperbolic spaces $\mathbb{H}^n$

Some data

- Ball model : $\mathbb{H}^n = \{\, x \in \mathbb{R}^n \mid \|x\| < 1 \,\}$
- Riemannian metric : $ds^2 = \big(\frac{1-\|x\|^2}{2}\big)^{-2} \, \|dx\|^2$
- Hyperbolic distance : $d(x,0) = \log \frac{1+\|x\|}{1-\|x\|}$
- Riemannian volume : $dV(x) = \big(\frac{1-\|x\|^2}{2}\big)^{-n} \, dx_1 \dots dx_n$
- Laplacian :

$$\Delta = \big(\tfrac{1-\|x\|^2}{2}\big)^2 \sum\nolimits_{j=1}^{n} \big(\tfrac{\partial}{\partial x_j}\big)^2 + (n-2)\, \tfrac{1-\|x\|^2}{2} \sum\nolimits_{j=1}^{n} x_j \, \tfrac{\partial}{\partial x_j}$$

Remarks

- Continuous analogs of homogeneous trees
- Similar large scale analysis + local analysis

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Large time estimates on $\mathbb{H}^n$

Large time kernel estimate

Assume that $1 < \varkappa < 2$. Then

$$|k_t^{\varkappa}(r)| \lesssim \begin{cases} |t|^{-\frac{3}{2}}(1+r)\,e^{-\frac{n-1}{2}r} & \text{if } |t| \geq 1+r \\ (1+r)^N e^{-\frac{n-1}{2}r} & \text{if } 1 \leq |t| \leq 1+r \end{cases}$$

Large time dispersive estimate

Let $1 < \varkappa \leq 2$ and $2 < q \leq \infty$. Then

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for large $|t|$

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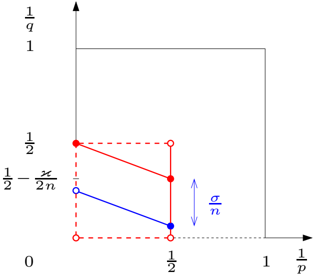
Strichartz inequality on $\mathbb{H}^n$

Strichartz inequality

Assume that $1 < \varkappa \leq 2$. Then

$$\|u(t,x)\|_{L^p_t H^{-\sigma,q}_x} \lesssim \|f\|_{L^2} + \|F(t,x)\|_{L^{\tilde{p}'}_t H^{\tilde{\sigma},\tilde{q}'}_x}$$

Here $(\frac{1}{p},\frac{1}{q})$, $(\frac{1}{\tilde{p}},\frac{1}{\tilde{q}})$ are admissible pairs
and $\sigma \geq \max\{0, n(\frac{1}{2}-\frac{1}{q})-\frac{\varkappa}{p}\}$, $\tilde{\sigma} \geq \max\{0, n(\frac{1}{2}-\frac{1}{\tilde{q}})-\frac{\varkappa}{\tilde{p}}\}$



Thank you for your attention

Dziękuję za uwagę