Convex log-Sobolev inequalities on the real line

Based on joint work with Yan Shu
(Université Paris Ouest Nanterre La Défense).

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Convex log-Sobolev inequality

μ – probability measure on $\mathbb{R}^n$ (s.t. $\mathbb{E}_\mu e^{s|x|} < \infty \quad \forall s > 0$)

Definition (convex LSI)

For every smooth, convex, Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Ent}_\mu(e^f) \leq C\, \mathbb{E}_\mu |\nabla f|^2 e^f.$$

(Recall: $\mathrm{Ent}_\mu(e^f) = \mathbb{E}_\mu fe^f - \mathbb{E}_\mu e^f \ln(\mathbb{E}_\mu e^f)$.)

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$$\mathbb{P}_{\mu^{\otimes N}}(f \ge \mathbb{E}_{\mu^{\otimes N}} f + t) \le \exp(-t^2/C).$$

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Questions

Which measures? Lower tail: $\mathbb{P}_{\mu^{\otimes N}}(f \le \mathbb{E}_{\mu^{\otimes N}} f - t) \le \dots$?

Main result: characterization on the real line

$F_\mu, \ F_{\exp} = $ CDFs of $\mu, \ \frac{1}{2}e^{-|x|}dx, \qquad U_\mu = F_\mu^{-1} \circ F_{\exp}$

Theorem (Shu, St., 2017)

Equivalent:

(i) μ on $\mathbb{R}$ satisfies convex LSI,

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More generally (under some assumptions on H),

$$\mathrm{Ent}_\mu(e^f) \leq \mathbb{E}_\mu\, H(f')e^f \quad \leftrightsquigarrow \quad U_\mu(x+h) - U_\mu(x) \leq \frac{1}{b}(H^*)^{-1}(1+h).$$

Tool: weak transportation inequalities

- Transport cost: $\mathcal{T}_2(\mu_1, \mu_2) = \inf_{X \sim \mu_1, Y \sim \mu_2} \mathbb{E}|X - Y|^2$.
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Ingredients of the proof

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$$\overline{\mathbf{T}}_2^- \iff \forall f \text{ convex, bdd below } \mathbb{E}_\mu\, e^{Q_1 f} e^{-\mathbb{E}_\mu f} \leq 1,$$
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- Control

$$\frac{1}{\mu((x,\infty))}\int_x^\infty \exp\Big((k(y-x)-1)_-^2\Big)d\mu(y) \le K.$$

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Concentration

For convex, 1-Lipschitz $f : \mathbb{R}^N \to \mathbb{R}$,

$$\mathbb{P}_{\mu^{\otimes N}}(|f - \mathbb{E}_{\mu^{\otimes N}} f| \ge t) \le 2\exp(-t^2/C').$$

Some open questions

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2. Convex LSI in $\mathbb{R}^n$?
3. *Concave* LSI? ($\overline{\mathbf{T}}_2^+$?)
4. Other functional inequalities?

Thank you for your attention.

Questions?

Yan Shu, M.St. *A characterization of a class of convex log-Sobolev inequalities on the real line*, https://arxiv.org/abs/1702.04698