On KLS conjecture for certain classes of convex sets

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Poincaré inequality

We say that a probability measure μ satisfies the Poincaré inequality if

$$\mathrm{Var}_\mu f = \int f^2 d\mu - \left(\int f d\mu\right)^2 \leq C_\mu \int |\nabla f|^2 d\mu.$$

Poincaré (spectral gap) constant = best value of C_μ

Motivating problem: KLS conjecture

Conjecture

Kannan, Lovasz, Simonovits (1995). There exists a universal number c such that for every uniform distribution μ on a convex body K satisfying

$$\mathbb{E}_\mu x_i = 0, \; \mathbb{E}_\mu (x_i x_j) = \delta_{ij}.$$

one has

$$C_K := C_\mu \leq c.$$

We call such bodies **isotropic**.

Equivalently:

there exists a universal number c such that for every uniform distribution μ on a convex body K

$$C_K \leq c \cdot C_K^{lin},$$

where

$$C_K^{lin} = \sup_{\text{linear } f} \frac{Var_\mu(f)}{\int_K |\nabla f|^2 d\mu}.$$

Other related conjectures and results

1) **Hyperplane conjecture.** (**Bourgain**). There exists universal $c > 0$ such that for any convex set $K \subset \mathbb{R}^d$ of $\mathrm{Vol(K)} = 1$ there exists a hyperplane L such that $\mathrm{Vol(K \cap L) > c}$.
2) **Thin-shell conjecture.** This is KLS conjecture for concrete function $f = |X|$. Can be put in the following way:

$$\mathbb{E}_\mu((|X| - \sqrt{d})^2) \le c$$

   for some universal constant c and every uniform distibution μ on an isotropic convex body $K \subset \mathbb{R}^d$.
3) **Central limit theorem for convex bodies**. Estimates of the thin-shell type $\Longrightarrow$ central limit theorem for convex bodies (initiated by **Sudakov** (70's), recent result: **Klartag, 2006**).

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What is known?

1. Special bodies

l_p-balls

$$B_p = \{x : |x_1|^p + |x_2|^p + \cdots + |x_d|^p \leq 1\}, \; 1 \leq p \leq \infty.$$

It is known that

$$C_{B_p} \leq \frac{c(p)}{d^{\frac{2}{p}}}$$

and B_p satisfies the KLS conjecture.
Case $1 \leq p \leq 2$: S. Sodin [2008]
Case $2 \leq p \leq \infty$: R. Latała, J. O. Wojtaszczyk [2008]
Simplex: F. Barthe, P. Wolff [2009].

2. Unconditional convex bodies

Unconditional bodies are bodies which are invariant with respect to the mappings

$$x \to (\pm x_1, \ldots, \pm x_d).$$

K. Klartag [2009]: KLS conjecture holds for unconditional convex bodies up to the logarithmic factor

$$C_K \leq C \log d.$$

3. General bodies

O. Guédon, E. Milman [2011] and **R. Eldan** [2013] :

$$C_K \leq Cd^{\frac{1}{3}} \log d.$$

Best known result:

Y.T. Lee, S. Vempala [2016]

$$C_K \leq Cd^{\frac{1}{4}}.$$

Main result

Level set

$$K_E = \{V \leq E\}.$$

Theorem

Let $\mu = e^{-V}dx$ be a log-concave probability measure with $\min(V) = 0$. There exists a universal constant $C > 1$ and $E \leq d$ such that

$$\frac{1}{C} \leq Vol^{\frac{1}{d}}(K_E) \leq C$$

and

$$C_{K_E} \leq C \cdot C_\mu \log(e + \sqrt{d}\, C_\mu).$$

Removing logarithmic factor

Theorem

Let $V_i\colon \mathbb{R} \mapsto \mathbb{R}$ be convex functions, $1 \leq i \leq d$. Assume that $\min(V_i) = 0$ and every $\mu_i = e^{-V_i} dx_i$ is a probability measure witn barycenter at the origin. There exists a universal constant $C > 1$ and $E \leq d$ such that

$$\frac{1}{C} \leq Vol^{\frac{1}{d}}(K_E) \leq C$$

and

$$C_{K_E} \leq C \log\big(e + \min(A_2, d)\big) C_{K_E}^{lin},$$

where

$$A_2 = \frac{1}{\sqrt{d}} \|(\alpha_i)\|,$$

$$\alpha_i = \inf\Big\{ t > 0\colon \int_{\mathbb{R}} e^{\frac{|V_i'(y)y-1|}{t}} dy \leq e \Big\}.$$

Example

Let $p_i^{\pm} \in [1, P]$, $i = 1, \dots, d$, for a fixed arbitrary constant $P \geq 1$. Then there exists $E \leq d$ so that the generalized Orlicz ball:

$$K_E := \Big\{ x \in \mathbb{R}^d \; ; \; \sum_{i=1}^{d} \big((x_i)_+^{p_i^+} + (x_i)_-^{p_i^-} \big) \leq E \Big\},$$

satisfies

$$\frac{1}{C} \leq Vol^{\frac{1}{d}}(K_E) \leq C$$

and

$$C_{K_E} \leq C \log(e + \min(P, d)) C_{K_E}^{lin}$$

for some universal $C > 0$.

Proof. Step 1. From μ to linearized measure on annulus

Probability measure

$$\mu = e^{-V} dx.$$

Generalized ball

$$K_E = \{V \le E\}.$$

Linearized probability measure

$$\mu_{K_E} = \frac{1}{Z_E} e^{-\left(E+n(\|x\|_{K_E}-1)\right)} dx$$

Linearized probability measure on annulus

$$\mu_{K_{E,\omega}} = \frac{1}{Z_{E,\omega}} e^{-\left(E+n(\|x\|_{K_E}-1)\right)} I_{1-\omega\le\|x\|_{K_E}\le 1} dx.$$

Ratio of normalizing constants

Note that $\|X_{K_E}\|$ has Gamma distribution with respect to μ_{K_E}. In particular

$$Z_E = \frac{d!e^d}{d^d} e^{-E} Vol(K_E),$$

$$\frac{Z_{E,\omega}}{Z_E} = \frac{d^d}{(d-1)!} \int_{1-\omega}^{1} e^{-dr} r^{d-1} dr \geq c\sqrt{d}\omega$$

provided $0 \leq \omega \leq \frac{1}{\sqrt{d}}$.

Uniform estimate

Corollary

$$\frac{d\mu_{K_{E,\omega}}}{d\mu} \leq \frac{e^{d\omega}}{c\sqrt{d}\omega Z_E}$$

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We apply estimate $\frac{d\mu_{K_{E,\omega}}}{d\mu} \leq \frac{e^{d\omega}}{c\sqrt{d}\omega Z_E}$ and the following result

Theorem

[Barthe–Milman] *Let ν_1, ν_2 be probability measures. Assume that $\|\frac{d\nu_2}{d\nu_1}\|_{L^p(\nu_1)} \leq L$ for some $p \in (1, \infty]$. Then setting $q = p^* = \frac{p}{p-1}$, we have:*

$$K_2(r) \leq 2LK_1^{1/q}(r/2) \quad \forall r > 0,$$

where K_1, K_2 are concentration functions of ν_1, ν_2.

Thus we transfer concentration estimates from μ to $\mu_{E,K}$.

Step 2. From annulus to cone measure

Note that the mapping

$$T(x) = \frac{x}{\|x\|_{K_E}}$$

pushes forward the annulus measure $\mu_{K_{E,\omega}}$ onto the cone measure

$$\sigma = \frac{1}{Vol(K_E)} \frac{\langle x, \nu_{\partial K_E} \rangle}{d} \cdot \mathcal{H}^{d-1}|_{\partial K_E}.$$

To transfer concentration from the annulus measure to the cone measure we apply

Lemma

[E. Milman] *For every* 1*-Lipshitz function* f

$$\int |f - med_{\mu_2} f| d\mu_2 \leq \int |f - med_{\mu_1} f| d\mu_1 + W_1(\mu_1, \mu_2),$$

where W_1 *is the Kantorovich (Wasserstein) distance.*

and

Lemma

$$W_1(\mu_{K_{E,\omega}}, \sigma) \leq \frac{d+1}{d} \omega \int \|x\|_{K_E} d\lambda_{K_E},$$

where λ_{K_E} *is the Lebesgue probability measure on* K_E*.*

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Step 3. From cone measure to uniform measure

Hardy-type inequality. For any convex Ω

$$\int_\Omega |f - med_{\lambda_\Omega}| d\lambda_\Omega \leq \int_{\partial\Omega} |f - med_{\sigma_\Omega}| d\sigma_\Omega + \frac{1}{d} \int |\nabla f||x| d\lambda_\Omega,$$

where σ_Ω is the corresponding cone measure.
Proof:

$$\begin{aligned} d \int_\Omega g dx &= \int_\Omega \mathrm{div}(x) g dx = - \int_\Omega \langle x, \nabla g \rangle dx + \int_{\partial\Omega} \langle x, \nu_\Omega \rangle g d\mathcal{H}^{d-1} \\ &\leq \int_\Omega |x||\nabla g| dx + \int_{\partial\Omega} \langle x, \nu_\Omega \rangle g d\mathcal{H}^{d-1}. \end{aligned}$$

Setting $g = |f - med_{\sigma_\Omega}|$ we complete the proof.

Last step

Collecting everything together we get that for every 1-Lipschitz function f

$$\int_{\Omega} |f - med_{\lambda_\Omega}| d\lambda_\Omega \leq C(\omega, d, Z_E, C_\mu).$$

Optimize parameters: choosing $\omega = \frac{1}{d}$ and applying lemma below we get the result.

Lemma

Let $\mu = e^{-V}dx$ be a log-concave probability measure with $\min(V) = 0$. *Set*

$$Level(V) = \left\{E \geq 0 : e^{-E} Vol(K_E) \geq \frac{1}{e}\frac{d^d e^{-d}}{d!}\right\}.$$

Then $Level(V) = [E_{min}, E_{max}]$ is a non-empty interval such that

1. $$E_{min} \leq d$$
2. $$1 \leq E_{min} - E_{max} \leq e\sqrt{2\pi d}(1 + o(1))$$
3. $$c(d) \leq Vol^{\frac{1}{d}}(E_{min}) \leq Vol^{\frac{1}{d}}(E_{max}) \leq ec(d)(1 + o(1)),$$

   where $c(d) \to 1$ as $d \to \infty$.

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