# Hamilton-Jacobi equation in metric spaces and applications

Cyril Roberto

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# The Classical Hamilton-Jacobi (HJ) Equation

The solution (in some sense) of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t u(t,x) = -\frac{1}{2}|\nabla_x u(t,x)|^2 & t > 0, x \in \mathbb{R}^n \\ u(0,x) = f(x) & x \in \mathbb{R}^n \end{cases}$$

is given by the celebrated Hopf-Lax formula

$$u(t,x) = Q_t f(x) := \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t}|x-y|^2 \right\}$$

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- $|(x_1, \ldots, x_n)| := \sqrt{\sum x_i^2}$ is the Euclidean distance
- $(Q_t)_{t \geqslant 0}$ is a semi-group: $Q_{t+s} = Q_t(Q_s)$ also called inf-convolution operator

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For example if $f$ is Lipschitz, i.e. $Lip(f) := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty$, then

- $(t,x) \mapsto Q_t f(x)$ is Lipschitz (on $[0,\infty) \times \mathbb{R}^n$)
- $Q_0 f(x) = f(x)$ for all $x \in \mathbb{R}^n$
- Everywhere where this makes sense (so a.e. in $(t,x)$ by Rademacher) $Q_t f(x)$ satisfies $\partial_t Q_t f(x) = -\frac{1}{2}|\nabla_x Q_t f(x)|^2$

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One can replace $\frac{1}{2}|\cdot|^2$ by a convex function $L$ in the Hopf-Lax formula (plugging $tL(|x-y|/t)$). It becomes a solution of the HJ Equation

$$\partial_t u(t,x) = -L^*(|\nabla_x u(t,x)|)$$

with $L^*(x) = \sup_y \{xy - L(y)\}$.

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# Hamilton Jacobi Equation In Metric Space - setting

- $(X, d)$ is a metric space, complete, separable and for which balls are compact.
- If for all $x, y \in X$ there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and for all $s, t \in [0,1]$, $d(\gamma(s), \gamma(t)) = |t - s| d(x, y)$, then $X$ is called a geodesic space.

For $f\colon X \to \mathbb{R}$, define the length of the gradient

$$|\nabla^- f|(x) := \begin{cases} 0 & \text{if } x \text{ is an isolated point} \\ \limsup_{y \to x} \frac{[f(y) - f(x)]_-}{d(x,y)} & \text{otherwise} \end{cases}$$

where $[A]_- := \max(-A, 0)$.

- If $f$ is Lipschitz then $|\nabla^- f|(x) \leqslant Lip(f) < \infty$
- If $X$ is a Riemannian manifold $|\nabla^- f|(x)$ coincides with the length of the vector $\nabla f(x)$ in the tangent space at $x$.

# Main Theorem

**Theorem (Ambrosio-Gigli-Savaré '14, Gozlan-R-Samson '14)**

*Let $f\colon X \to \mathbb{R}$ be continuous and bounded. Then*

$$\begin{cases} \frac{d}{dt_+} Q_t f(x) \leqslant -\frac{1}{2}|\nabla^- Q_t f|(x)^2 & \forall t > 0, \quad \forall x \in X \\ \frac{d}{dt_+} Q_t f(x)_{|t=0} \geqslant -\frac{1}{2}|\nabla^- f|(x)^2 & t = 0, \quad \forall x \in X \end{cases}$$

*where $Q_t f(x) := \inf_{y \in X} \left\{ f(y) + \frac{1}{2t} d(x,y)^2 \right\}$. $\lim_{t\to 0} Q_t f = f$.*
*If in addition $X$ is geodesic, then $\frac{d}{dt_+} Q_t f(x) = -\frac{1}{2}|\nabla^- Q_t f|(x)^2$ for all $t \geqslant 0$ and all $x \in X$.*

- Lott-Villani '07 if $(X, d, \mu)$ is a compact measured geodesic space with $\mu$ satisfying a local Poincaré inequality + doubling condition. Result holds for all $t$ and for $\mu$-almost all $x \in X$. Result holds for all $t$ and all $x$ under some additional curvature condition + finite dim.
- If $X$ is a Riemannian manifold, Villani '09 get the result for all $t, x$.

# Where The Audience Is Asked To Choose Between Application I and II

- Application I : Otto-Villani's theorem (in metric spaces).
- Application II : Concentration versus the Poincaré Inequality (in metric spaces).

# Application I: Introduction - Talagrand's inequality

Given two probability measures $\nu_1, \nu_2$ on $X$, set

$$T_2(\nu_1, \nu_2) := \inf_{\pi} \iint \frac{1}{2} d(x,y)^2 \pi(dx, dy)$$

where the inf runs over all couplings (i.e. prob. meas. on $X \times X$) $\pi$ with first marginal $\nu_1$ and second marginal $\nu_2$.
A probability measure $\mu$ satisfies the Talagrand transport-entropy inequality (with constant $C$) if

$$T_2(\nu, \mu) \leqslant C H(\nu|\mu) \qquad \forall \nu \text{ prob. meas.}$$

where $H(\nu|\mu) := \begin{cases} \int \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} d\mu & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise} \end{cases}$ is the relative entropy of $\nu$ with respect to $\mu$.
Ex: For $\mu = \gamma_n$ the standard Gaussian on $X = \mathbb{R}^n$ satisfies the inequality with $C = 1$ (optimal) Talagrand '96.

# Application I: Transport and inf-convolution operator

Recall Talagand's inequality: $T_2(\nu, \mu) \leqslant CH(\nu|\mu)$ for all $\nu$.

**Theorem (Bobkov-Gotze '99)**

*$\mu$ satisfies the Talagrand transport-entropy inequality with constant $C$ iff for all $f \colon X \to \mathbb{R}$ continuous and bounded*

$$\int e^{\frac{Q_1 f}{C}} d\mu \leqslant e^{\frac{1}{C}\int f d\mu}$$

*where $Q_1 f(x) = \inf_y \{f(y) + \frac{1}{2} d(x, y)^2\}$.*

# Application I: log-Sobolev inequality

$\mu$ satifies the log-Sobolev inequality, with constant $C$, if for all $f\colon X \to \mathbb{R}$, say Lipschitz,

$$\mathrm{Ent}_\mu(e^f) \leqslant \frac{C'}{2} \int |\nabla^- f|^2 e^f d\mu$$

where

$\mathrm{Ent}_\mu(g) := \int g d\mu H\left(\frac{g d\mu}{\int g d\mu} | \mu\right) = \int g \log g d\mu - \int g d\mu \log \int g d\mu.$

Ex: if $\mu = \gamma_n$ and $X = \mathbb{R}^n$ then $C' = 1$ (optimal) Stam '59, Gross '75. Applications:

- Hypercontractivity Gross '75.
- Concentration Herbst 70s.
- Gaussian Isoperimetry, Bakry-Ledoux '96.

# Application I: Otto-Villani's Theorem

Talagrand: $T_2(\nu,\mu) \leqslant CH(\nu|\mu)$
Log-Sobolev: $\mathrm{Ent}_\mu(e^f) \leqslant \frac{C'}{2}\int |\nabla^- f|^2 e^f d\mu$

**Theorem (Gozlan-R-Samson '14)**

*Assume that $\mu$ satisfies the log-Sobolev inequality with some constant $C'$. Then $\mu$ satisfies the Talagrand transport-entropy Inequality with constant $C \leqslant C'$.*

- Otto-Villani '00.
- Bobkov-Gentil-Ledoux '01, alternative proof based on HJ .
- Extensions in many directions Gentil-Guillin-Miclo '05, Wang '04 (path spaces), Lott-Villani '07 (geodesic spaces), Balogh-Engoulatov-Hunziker-Maasalo '09.
- In the general setting of metric spaces Gozlan '09 (Large deviation), Gozlan-R-Samson '12 (tensorisation), Gigli-Ledoux '11 (Gradient flow).
- Talagrand $\not\Rightarrow$ log-Sob, Cattiaux-Guillin' 06.

# Application I: Otto-Villani's Theorem

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# Proof of Otto-Villani's Theorem

Recall Bobkov-Gotze dual chracterization: $\int e^{\frac{Q_1 f}{C}} d\mu \leqslant e^{\frac{1}{C}\int f d\mu}$
We will proove better, namely that

$$\|e^{Q_t f}\|_{a+\frac{t}{C'}} \leqslant \|e^f\|_a \qquad \forall t, a \geqslant 0, \forall f$$

In fact if this is true, then choose $t = 1$ and send $a$ to 0 to obtain

$$\left(\int e^{\frac{Q_1 f}{C'}} d\mu\right)^{C'} \leftarrow \|e^{Q_1 f}\|_{a+\frac{1}{C'}} \leqslant \|e^f\|_a \rightarrow e^{\int f d\mu}$$

which is precisely the dual characterization of Talagrand transport-entropy inequality.

# Proof of Otto-Villani's Theorem

In order to proove

$$\|e^{Q_t f}\|_{a+\frac{t}{C'}} \leqslant \|e^f\|_a \qquad \forall t, a \geqslant 0, \forall f$$

we set $F(t) := \log \|e^{Q_t f}\|_{k(t)}$ with $k(t) = a + \frac{t}{C'}$ and differentiate:

$$F'(t) = \frac{k'(t)}{k(t)} \frac{1}{\int e^{k(t)Q_t f} d\mu} \left[ \mathrm{Ent}_\mu(e^{k(t)Q_t f}) + \frac{k^2(t)}{k'(t)} \int \frac{d}{dt_+} Q_t f e^{k(t)Q_t f} d\mu \right]$$

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$$\leqslant \frac{k'(t)}{k(t)} \frac{1}{\int e^{k(t)Q_t f} d\mu} \left[ \mathrm{Ent}_\mu(e^{k(t)Q_t f}) - \frac{1}{2k'(t)} \int |\nabla^- k(t) Q_t f|^2 e^{k(t)Q_t f} d\mu \right]$$

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$$\leqslant \frac{k'(t)}{k(t)} \frac{1}{\int e^{k(t)Q_t f} d\mu} \left[ \mathrm{Ent}_\mu(e^{k(t)Q_t f}) - \frac{C'}{2} \int |\nabla^- k(t) Q_t f|^2 e^{k(t)Q_t f} d\mu \right]$$

The log-Sobolev inequality $\mathrm{Ent}_\mu(e^g) \leqslant \frac{C'}{2} \int |\nabla^- g|^2 e^g d\mu$ applied to $g = k(t)Q_t f$ guarantees that the r.h.s. is non positive, hence $F$ is non-increasing and therefore

$$\|e^{Q_t f}\|_{a+\frac{t}{C'}} = e^{F(t)} \leqslant e^{F(0)} = \|e^f\|_a,$$

ending the proof.

# Application II: Introduction (Concentration)

- Define on $X^n$ the $\ell_2$-product distance

$$d^{(n)}(x,y) := \left(\sum_{i=1}^{n} d(x_i,y_i)^2\right)^{1/2}$$

- Given a Borel set $A \subset X^n$, we define its enlargement as

$$A_r^{(n)} := \{x \in X : d^{(n)}(x,A) \leqslant r\} \qquad \text{Notation } A_r = A_r^{(1)}$$

- A probability measure $\mu$ on $X$ satisfies a *concentration inequality*, with concentration profile $\alpha$ (non-increasing), if

$$\mu\left(A_r\right) \geqslant 1 - \alpha(r) \qquad \forall A \subset X \text{ with } \mu(A) \geqslant 1/2,$$

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- A probability measure $\mu$ on $X$ satisfies a dimension free *concentration inequality*, with concentration profile $\alpha$ (non-increasing), if

$$\mu^n\left(A_r^{(n)}\right) \geqslant 1-\alpha(r) \qquad \forall A \subset X^n \text{ with } \mu^n(A) \geqslant 1/2, \quad \forall n$$

We denote by $CI_2(\alpha)$ and $CI_2^\infty(\alpha)$ respectively such properties.

# Application II: Introduction (The Poincaré Inequality)

A probablity measure $\mu$ on $X$ satisfies the Poincaré Inequality if there exists $\lambda > 0$ such that for all $f$ Lipschitz it holds

$$\int f^2 d\mu \; - \left( \int f d\mu \; \right)^2 =: \mathrm{Var}_\mu \; (f) \leqslant \frac{1}{\lambda} \int |\nabla^- f|(x)^2 d\mu$$

Ex: if $X = \mathbb{R}^n$ and $\mu = \gamma_n$ the standard Gaussian measure, then $\lambda = 1$ (optimal).

**Theorem (Gromov-Milman '83)**

*Assume that $\mu$ satisfies a Poincaré inequality with constant $\lambda > 0$. Then $IC_2^\infty(\alpha)$ holds with $\alpha(r) := be^{-a\sqrt{\lambda}r}$, $r \geqslant 0$:*

$$\mu^n \left( A_r^{(n)} \right) \geqslant 1 - be^{-a\sqrt{\lambda}r} \qquad \forall A \subset X^n \text{ with } \mu^n(A) \geqslant 1/2, \quad \forall n$$

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# Poincaré versus dimension free concentration.

$CI_2^\infty(\alpha)$: $\mu^n(A_r^{(n)}) \geqslant 1-\alpha(r)$, for all $A$ with $\mu^n(A) \geqslant 1/2$, $\forall r, n$. Poincaré: $\mathrm{Var}_\mu(f) \leqslant \frac{1}{\lambda}\int |\nabla^- f|(x)^2 d\mu$.

**Theorem (Gozlan-R-Samson '15)**

*If $\mu$ satisfies $CI_2^\infty(\alpha)$, then the Poincaré Inequality holds with $\lambda = \sup_{r>0:\alpha(r)\leqslant\frac{1}{2}} \left\{\frac{\bar\Phi^{-1}(\alpha(r))}{r}\right\}^2$ where $\bar\Phi(x) = \int_x^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}}du$.*

i.e. $CI_2^\infty(\alpha) \Leftrightarrow$ Poincaré (for non trivial $\alpha$: note that $\bar\Phi^{-1}(\frac{1}{2}) = 0$).

- (self-improvement. Talagrand '91) If $\mu$ satisfies $CI_2^\infty(\alpha)$ with a "minimal" profile $\alpha = \frac{1}{2}\mathbb{1}_{[0,r_o]} + a_o\mathbb{1}_{(r_o,\infty)}$, $r_o > 0$, $a_o \in (0, \frac{1}{2})$, then $\mu$ satisfies $CI_2^\infty(\alpha)$ with $\alpha(r) = be^{-ar}$.
- In case of the Gaussian measure $\gamma_n$, $CI_2^\infty(\bar\Phi)$ holds so that the theorem leads to $\lambda = 1$ which is optimal.
- If $\alpha(r)$ goes too fast to 0 (such as $be^{-ar^k}$ with $k > 2$) then $\lambda = +\infty$ and $\mu$ is a Dirac mass.
- Extension to the convex situation. See also Adamczak-Strzelecki '17, Gozlan-R-Samson-Shu-Tetali '15, Feldheim-Marsiglietti-Nayar-Wang '15.

# Proof. A key Lemma: concentration and inf-convolution operator

**Lemma (Gozlan-R-Samson '11)**

*The following are equivalent*

$(i)$ *$CI_2(\alpha)$ holds:* $\mu(A_r) \geqslant 1 - \alpha(r)$ *for all* $A \subset X$*, all* $r \geqslant 0$.

$(ii)$ *For any* $f \colon X \to \mathbb{R} \cup \{+\infty\}$ *bounded below and such that* $\mu(\{f = +\infty\}) < \frac{1}{2}$ *($m(f)$ being a median of $f$)*

$$\mu(Q_t f > m(f) + r) \leqslant \alpha(\sqrt{2rt}) \qquad \forall r, t > 0.$$

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$$\mu(Q_t f > m(f) + r) \leqslant \alpha(\sqrt{2rt}) \qquad \forall r, t > 0.$$

- Let $h : X \to \mathbb{R}$ be bounded, Lipschitz with $\int h d\mu = 0$.
  Set $f_n(x) = h(x_1) + \cdots + h(x_n), \qquad x = (x_1, \ldots, x_n) \in X^n$.
- Apply $(ii)$ of the Lemma to $\mu^n$, $f_n$, $t = \frac{1}{\sqrt{n}}$ and $r = \frac{\sqrt{n}u}{2}$

$$\mu^n \left( Q^{(n)}_{\frac{1}{\sqrt{n}}} f_n > m(f_n) + \frac{\sqrt{n}u}{2} \right) \leqslant \alpha(\sqrt{u})$$

with $Q^{(n)}_t f_n(x) = \inf\{f_n(y) + \frac{1}{2t} \sum_1^n d^2(x_i, y_i)\} = \sum_1^n Q_t h(x_i)$.

# Proof

$$\mu^n\left(Q^{(n)}_{\frac{1}{\sqrt{n}}} f_n > m(f_n) + \frac{\sqrt{n}u}{2}\right) \leqslant \alpha(\sqrt{u}) \text{ and } Q^{(n)}_t f_n = \sum Q_t h(x_i).$$

Set $\sigma_n := \sqrt{\mathrm{Var}_\mu(Q_{\frac{1}{\sqrt{n}}} h)}$.

$$\mu^n\left(Q^{(n)}_{\frac{1}{\sqrt{n}}} f_n > m(f_n) + \frac{\sqrt{n}u}{2}\right) = \mu^n\left(\frac{\sum_{i=1}^n \left[Q_{\frac{1}{\sqrt{n}}} h(x_i) - \int Q_{\frac{1}{\sqrt{n}}} h d\mu\right]}{\sqrt{n}\sigma_n} > \frac{m(f_n)}{\sqrt{n}\sigma_n} + \frac{u}{\sigma_n} + \frac{\sqrt{n}}{2\sigma_n} \int h - Q_{\frac{1}{\sqrt{n}}} h d\mu\right)$$

# Proof

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Observe that, in the limit $n \to \infty$, $\sigma_n \to \sqrt{\mathrm{Var}_\mu(h)}$, $\frac{m(f_n)}{\sqrt{n}} \to 0$
$\sqrt{n}[h(x) - Q_{\frac{1}{\sqrt{n}}} h(x)] \to -\frac{d}{dt_+} Q_t h(x)|_{t=0} \leqslant \frac{1}{2}|\nabla^- h|(x)^2$ by HJ.
By the CLT we get

$$\bar{\Phi}\left(\frac{u}{\mathrm{Var}_\mu(h)} + \frac{\int |\nabla^- h|^2 d\mu}{4\mathrm{Var}_\mu(h)}\right) \leqslant \alpha(\sqrt{u})$$

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We obtained $\bar{\Phi}\left(\frac{u}{\mathrm{Var}_\mu(h)} + \frac{\int |\nabla^- h|^2 d\mu}{4\mathrm{Var}_\mu(h)}\right) \leqslant \alpha(\sqrt{u})$.

Given $u$ so that $\alpha(\sqrt{u}) < 1/2$, we have, for $k(u) := \bar{\Phi}^{-1}(\alpha(\sqrt{u}))$

$$k(u)\sqrt{\mathrm{Var}_\mu(h)} \leqslant u + \frac{1}{4}\int |\nabla^- h|^2 d\mu$$

which, applied to $sh$ implies after squaring

$$\mathrm{Var}_\mu(h) \leqslant \frac{1}{k^2(u)}\left( \quad \frac{u}{s} + \frac{s}{4}\int |\nabla^- h|^2 d\mu\right)^2$$

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# Open question

Consider $d_p^{(n)}(x,y) := (\sum_i d(x_i,y_i)^p)^{1/p}$, $p \geqslant 1$ and the corresponding enlargement $A_{r,p}^{(n)} := \{x \in X^n : d_p^{(n)}(x,A) \leqslant r\}$ and the corresponding dimension free concentration inequality $CI_p^\infty(\alpha)$

$$\mu^n(A_{r,p}^{(n)}) \geqslant 1 - \alpha(r) \qquad \forall n, \forall A \text{ with } \mu^n(A) \geqslant 1/2.$$

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