Concentration for Coulomb gases

and Coulomb transport inequalities

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Probability and Analysis

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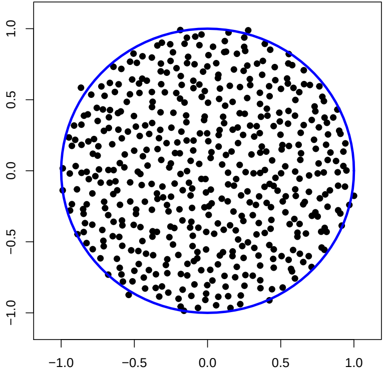
Motivation: concentration for Ginibre ensemble




```
plot(eig(randcg(500,500)/sqrt(500)))
```

Outline

Electrostatics

Coulomb gas model

Probability metrics and Coulomb transport inequality

Concentration of measure for Coulomb gases

Coulomb kernel in mathematical physics

- Coulomb kernel in $\mathbb{R}^d$, $d \geqslant 2$,

$$x \in \mathbb{R}^d \mapsto g(x) = \begin{cases} \log \dfrac{1}{|x|} & \text{if } d = 2, \\ \dfrac{1}{|x|^{d-2}} & \text{if } d \geq 3. \end{cases}$$

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- Fundamental solution of Poisson's equation

$$\Delta g = -c_d \, \delta_0 \quad \text{where} \quad c_d = \begin{cases} 2\pi & \text{if } d = 2, \\ (d-2)|\mathbb{S}^{d-1}| & \text{if } d \geq 3. \end{cases}$$

Coulomb energy and equilibrium measure

- Coulomb energy of probability measure μ on $\mathbb{R}^d$:

$$\mathscr{E}(\mu) = \iint g(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \in \mathbb{R} \cup \{+\infty\}.$$

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$$\mathcal{E}_V(\mu) = \mathcal{E}(\mu) + \int V(x)\mu(\mathrm{d}x).$$

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- If V is strong then μ_V is compactly supported with density

$$\frac{1}{2c_d}\Delta V$$

Examples of equilibrium measures

d	g	V	μ_V
1	2	$\infty \mathbf{1}_{\text{interval}^c}(x)$	arcsine
1	2	x^2	semicircle
2	2	$\lvert x \rvert^2$	uniform on a disc
$\geqslant 3$	d	$\lVert x \rVert^2$	uniform on a ball
$\geqslant 2$	d	radial	radial in a ring

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Coulomb gas or one component plasma

- Energy of N Coulomb charges in $\mathbb{R}^d$:

$$H_N(x_1,\ldots,x_N) = N\sum_{i=1}^{N} V(x_i) + \sum_{i\neq j} g(x_i - x_j)$$

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- Boltzmann–Gibbs measure $\mathbb{P}_{V,\beta}^N$ on $(\mathbb{R}^d)^N$:

$$\frac{\exp\left(-\frac{\beta}{2} H_N(x_1,\ldots,x_N)\right)}{Z_{V,\beta}}$$

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- $\mathbb{P}_{N,\beta}$ is neither product nor log-concave

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- Random empirical measure under $\mathbb{P}^N_{V,\beta}$:

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- Large Deviation Principle (Gozlan-C.-Zitt 2014)

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- Quantitative estimates? How to relate dist and $\mathscr{E}_V(\cdot)-\mathscr{E}_V(\mu_V)$?

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$$\mathrm{W}_p(\mu,\nu) = \inf_{\substack{(X,Y) \\ X\sim\mu,\, Y\sim\nu}} \mathbb{E}(|X-Y|^p)^{1/p}.$$

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Local Coulomb transport inequality

Theorem (Transport type inequality)

$$\mathrm{W}_1(\mu,\nu)^2 \leq C_D \mathscr{E}(\mu-\nu).$$

Local Coulomb transport inequality

Theorem (Transport type inequality)

$D \subset \mathbb{R}^d$ compact, $\mathrm{supp}(\mu+\nu) \subset D$, $\mathcal{E}(\mu) < \infty$ *and* $\mathcal{E}(\nu) < \infty$,

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- Extends Popescu local free transport inequality to any dim. d

Idea of proof of Coulomb transport inequality

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$$
\begin{aligned}
c_d \int f(x)(\mu - \nu)(\mathrm{d}x) &= -\int f(x)\Delta U^{\mu-\nu}(x)\mathrm{d}x \\
&\leq \|f\|_{\mathrm{Lip}} \left(|D_+| \int |\nabla U^{\mu-\nu}(x)|^2 \mathrm{d}x \right)^{1/2}
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$$= c_d \mathscr{E}(\mu - \nu).$$

- Finally $\mathrm{W}_1(\mu, \nu)^2 \leqslant |D_+| c_d \mathscr{E}(\mu - \nu)$.

Coulomb transport inequality for equilibrium measures

Theorem (Transport type inequality)

$$\mathrm{d}_{\mathrm{BL}}(\mu,\mu_V)^2 \leq C_{\mathrm{BL}}\Big(\mathscr{E}_V(\mu)-\mathscr{E}_V(\mu_V)\Big).$$

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For any probability measure μ on $\mathbb{R}^d$ with $\mathcal{E}(\mu) < \infty$

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Moreover if V has at least quadratic growth then

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- Growth condition is optimal for W_1

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If V does has reasonable growth then for every β, N, r

$$\mathbb{P}^N_{V,\beta}\left(\mathrm{d}_{\mathrm{BL}}(\mu_N, \mu_V) \geq r\right) \leqslant \mathrm{e}^{-a\beta N^2 r^2} \quad .$$

Moreover if V has at least quadratic growth then W_1 instead of d_{BL}.

- LDP shows that order in N is optimal

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$$\mathbb{P}^N_{V,\beta}\left(\mathrm{d}_{\mathrm{BL}}(\mu_N,\mu_V)\geq r\right)\leqslant \mathrm{e}^{-a\beta N^2r^2+\mathbf{1}_{d=2}(\frac{\beta}{4}N\log N)+b\beta N^{2-2/d}+c(\beta)N}.$$

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- See also Rougerie & Serfaty

Idea of proof of concentration

- Starting point

$$\mathbb{P}^N_{V,\beta}(\mathrm{W}_1(\mu_N,\mu_V) \geqslant r) = \frac{1}{Z^N_{V,\beta}} \int_{\mathrm{W}_1(\mu_N,\mu_V)\geqslant r} \mathrm{e}^{-\frac{\beta}{2} N^2 \mathcal{E}_V^{\neq}(\mu_N)} \mathrm{d}x.$$

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- Normalizing constant

$$\frac{1}{Z^N_{V,\beta}}\leqslant\exp\left\{N^2\frac{\beta}{2}\mathscr{E}_V(\mu_V)-N\left(\frac{\beta}{2}\mathscr{E}(\mu_V)-\mathrm{S}(\mu_V)\right)\right\}.$$

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- Regularization: g superharmonic, $\mu_N^{(\varepsilon)}=\mu_N*\lambda_\varepsilon$,

$$-N^2\mathcal{E}^{\neq}_V(\mu_N)\leqslant-N^2\mathcal{E}_V(\mu_N^{(\varepsilon)})+N\mathcal{E}(\lambda_\varepsilon)+N\sum_{i=1}^N(V*\lambda_\varepsilon-V)(x_i).$$

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- Coulomb transport $-\mathscr{E}_V(\mu_N^{(\varepsilon)})+\mathscr{E}_V(\mu_V)\leqslant-\frac{1}{C}\mathrm{W}_1^2(\mu_N^{(\varepsilon)},\mu_V)$.

Concentration for spectrum of Ginibre matrices

Corollary (Ginibre Random Matrices)

If M is $N \times N$ with iid Gaussian entries of variance $\frac{1}{N}$ in $\mathbb{C}$

- Eigenvalues of $M \propto \exp\left(-N\sum_{i=1}^{N}|x_i|^2\right)\prod_{i<j}|x_i - x_j|^2$

Concentration for spectrum of Ginibre matrices

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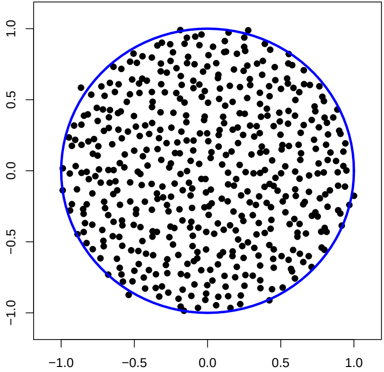
Concentration for spectrum of Ginibre random matrices




```
plot(eig(randcg(500,500)/sqrt(500)))
```

That's all folks!

Thank you for your attention

Dziękuję za uwagę

Exponential tightness

Theorem (Tightness)

For any $r \geq r_0$

$$\mathbb{P}^N_{V,\beta}(\mathrm{supp}(\mu_N) \not\subset B_r) = \mathbb{P}^N_{V,\beta}\Big(\max_{1\leq i\leq N} |x_i| \geq r\Big) \leq \mathrm{e}^{-cNV_*(r)},$$

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- Gives W_p versions of convergence and concentration

$$\mathrm{W}_p^p(\mu,\nu) \leqslant (2M)^{p-1}\mathrm{W}_1(\mu,\nu) \leqslant M(2M)^{p-1}\mathrm{d}_{\mathrm{BL}}(\mu,\nu).$$

For $p = 2$ we get $\mathbb{P}^N_{V,\beta}(\mathrm{W}_2(\mu_N,\mu_V) \geq r) \leqslant 2\mathrm{e}^{-cN^{3/2}r^2}$.

Convergence in Wasserstein distance

Corollary (Wasserstein convergence)

If V superquadratic and $\beta_N \geqslant \beta_V \frac{\log N}{N}$ then under $\mathbb{P}^N_{V,\beta_N}$ a.s.

$$\lim_{N\to\infty} \mathrm{W}_1(\mu_N, \mu_V) = 0.$$

Convergence at mesoscopic scale

Corollary (Mesoscopic convergence)

- *If $d = 2$ then*

$$\mathbb{P}^N_{V,\beta}\Big(\mathrm{d}_{\mathrm{BL}}\big(\tau^{N^s}_{x_0}\mu_N, \tau^{N^s}_{x_0}\mu_V\big) \geqslant CN^s\sqrt{\frac{\log N}{N}}\Big) \leqslant \mathrm{e}^{-cN\log N},$$

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Notes and comments

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