Free infinite divisibility for R-diagonal distributions

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18 V 2017

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- $\varphi(\cdot) = \mathbb{E}(\cdot)$.

Distribution of a random variable

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We will call a probability measure μ *the* **distribution** *of a self-adjoint random variable* $a \in \mathcal{A}$ *if*

$$\varphi\left(a^n\right) = \int_{\mathbb{R}} t^n \mu(dt).$$

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If a has distribution μ and b has distribution ν and a and b are free, then the distribution of $a + b$ is called the free convolution of μ and ν and is denoted by $\mu \boxplus \nu$.

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Infinite divisibility

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For a probability measure μ say that μ is $\boxplus$ infinitly divisible (resp. $\uplus$ infdiv.) if for any $n \geq 1$ there exists measure μ_n such that

$$\mu = \underbrace{\mu_n \boxplus \ldots \boxplus \mu_n}_{n-times}, \text{ resp. } \mu = \underbrace{\mu_n \uplus \ldots \uplus \mu_n}_{n-times},$$

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Theorem (Speicher)

Free (boolean) cumulants characterize free (boolean) independence. If elements are free (boolean) independent then respective mixed cumulants vanish.

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Free (boolean) cumulants characterize free (boolean) independence. If elements are free (boolean) independent then respective mixed cumulants vanish. For example if a_1, a_2 are free then $\kappa_k(a_{i_1}, \ldots, a_{i_k}) = 0$ whenever there are p, q such that $i_p \neq i_q$ where $i_j \in \{1, 2\}$.

R-diagonals

Power series

For a compactly supported probability measure μ, take $a \sim \mu$ and define two power series:

$$R_\mu(z) = \sum_{n=1}^{\infty} \kappa_n(a) z^n, \qquad \eta_\mu(z) = \sum_{n=1}^{\infty} \eta_n(a) z^n$$

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On the other hand all probability measures are infinitely divisible with respect to $\uplus$.

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There is a bijection $\mathbb{B}$ between $\uplus$ infinitely divisible measures (all measures) and $\boxplus$ infinitely divisible measures.

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Each sequences of boolean cumulants is a sequence of free cumulants of $\boxplus$-ID measure.

Distribution and infinite divisibility of non-normal elements

Definition

The $*$-distribution of a non-normal random variable a is a linear functional μ on the algebra $\mathbb{C}\langle z, z^*\rangle$ of polynomials of non-commuting variables z, z^*, defined by

$$\mu_{z,z^*}(P) = \varphi(P(a, a^*)) \qquad \forall\, P \in \mathbb{C}\langle a, a^*\rangle.$$

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Free (boolean) convolution of $*$-distributions is the $*$-distribution of $a+b$ where a has $*$-distribution μ and b has $*$-distribution ν and a, b are freely (boolean) independent and it is denoted by $\mu \boxplus \nu$ (resp. $\mu \uplus \nu$).

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BBP for non-normal elements

For a r.v. a with the $*$-distribution μ we can define

$$R_\mu(z, z^*) = \sum_{n=1}^{\infty} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{1,*\}} \kappa_n(a^{\varepsilon_1}, \ldots, a^{\varepsilon_n}) z^{\varepsilon_1} \cdots z^{\varepsilon_n},$$

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Theorem (Bercovici, Nica, Noyes, Sz.)

There is a bijection $\mathbb{B}_{(1,*)}$ between all $*$-distributions and $\boxplus$-ID $*$-distributions, for each $*$-distr. μ determined by

$$R_{\mathbb{B}_{(1,*)}(\mu)}(z, z^*) = \eta_\mu(z, z^*)$$

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Simplest example $\alpha_1 = \beta_1 = 1$ and $\alpha_n = \beta_n = 0$ for $n \geq 2$, is co called circular distribution, it is limit of Ginibre ensembles.

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- R-diagonality of $*$-distribution of a can be expressed in terms of $*$-moments of a.
- If a, b are free and both have R-diagonal $*$-distribution, then the $*$-distribution of ab is also R-diagonal.

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To study $\boxplus$-ID R-diagonals, we have to verify which sequences $(\alpha_n), (\beta_n)$ can appear in (1).

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Theorem (Bercovici, Nica, Noyes, Sz.)

For any two probability measures σ_1, σ_2 compactly supported on $[0, +\infty)$, there exists a unique η-diagonal $*$-distribution μ, such that if a has $*$-distribution μ, then aa^* and a^*a have distributions σ_1, σ_2.

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Theorem (Bercovici, Nica, Noyes, Sz.)

1) If a has $\boxplus$-ID R-diagonal $*$-distribution then distributions of aa^* and a^*a are $\boxplus$-ID.
2) If a, b are free and have $\boxplus$-ID R-diagonal $*$-distributions then the distribution of ab is $\boxplus$-ID R-diagonal.

Thank you for your attention!