Integrated density of states for subordinate Brownian motions on the Sierpiński gasket: existence and asymptotics

Katarzyna Pietruska-Pałuba
University of Warsaw

(joint with Kamil Kaleta, Dorota Kowalska)

Będlewo, May 18th, 2017

- Kamil Kaleta, Katarzyna Pietruska-Pałuba, *Integrated density of states for Poisson-Schrödinger perturbations of subordinate Brownian motions on the Sierpiński gasket.* Stochastic Process. Appl. 125 (2015), no. 4, 1244–1281.

- Kamil Kaleta, Katarzyna Pietruska-Pałuba, *Lifschitz singularity for subordinate Brownian motions in presence of the Poissonian potential on the Sierpiński gasket*, preprint, ArXiv:1406.5651.

- Dorota Kowalska, Katarzyna Pietruska-Pałuba, *Lifschitz tail and sausage asymptotics for stable processes in the Poissonian environment on the Sierpinski gasket*, preprint, ArXiv:1406.4970.

Integrated density of states – classical setting

What is the integrated density of states?

Integrated density of states – classical setting

What is the integrated density of states?

Consider the Brownian motion on $\mathbb{R}^d$, possibly with some random interaction (killing potential).

Integrated density of states – classical setting

What is the integrated density of states?

Consider the Brownian motion on $\mathbb{R}^d$, possibly with some random interaction (killing potential).

- Its generator is not a compact operator $\Rightarrow$ its spectrum is not discrete.

Integrated density of states – classical setting

What is the integrated density of states?

Consider the Brownian motion on $\mathbb{R}^d$, possibly with some random interaction (killing potential).

- Its generator is not a compact operator $\Rightarrow$ its spectrum is not discrete.
- Eigenvalues $\leftrightarrow$ possible energy levels of electrons.
- Pauli exclusion principle: one electron per energy level.

Integrated density of states – classical setting

What is the integrated density of states?

Consider the Brownian motion on $\mathbb{R}^d$, possibly with some random interaction (killing potential).

- Its generator is not a compact operator $\Rightarrow$ its spectrum is not discrete.
- Eigenvalues $\leftrightarrow$ possible energy levels of electrons.
- Pauli exclusion principle: one electron per energy level.
- How to count these energy levels?
- What to do in an 'infinite setting'? How to distribute countably many electrons on a continuum of spectral energies?

Random interaction with potential:

• take $V:\mathbb{R}^d \to \mathbb{R}_+$, measurable and regular enough (Kato class) then one can define an L^2-semigroup (better: C_0-semigroup if Kato) by means of the Feynman-Kac formula

$$P_t^V f(x) = \mathbb{E}_x[f(X_t)\mathrm{e}^{-\int_0^t V(X_s)\mathrm{d}s}],$$

Random interaction with potential:

• take $V : \mathbb{R}^d \to \mathbb{R}_+$, measurable and regular enough (Kato class) then one can define an L^2-semigroup (better: C_0-semigroup if Kato) by means of the Feynman-Kac formula

$$P_t^V f(x) = \mathbb{E}_x[f(X_t)\mathrm{e}^{-\int_0^t V(X_s)\mathrm{d}s}],$$

• can add killing on exiting an open set U :

$$P_t^{V,U} f(x) = \mathbb{E}_x[f(X_t)\mathrm{e}^{-\int_0^t V(X_s)\mathrm{d}s}\mathbf{1}\{\tau_U > t\}].$$

Without further assumptions, these semigroups are not trace-class and the spectrum of their generator is hard to analyze.

Integrated density of states

Remedy: restrict the system to a finite volume, Λ.

Integrated density of states

Remedy: restrict the system to a finite volume, Λ.

- Only finite number of electrons in a finite volume.

Integrated density of states

Remedy: restrict the system to a finite volume, Λ.

- Only finite number of electrons in a finite volume.

- Laplacian in the finite volume: set boundary values (choose: Dirichlet or Neumann – which corresponds to killing or reflecting), then the semigroup $P_t^\Lambda = \mathrm{e}^{-t\Delta}$ consists of trace-class operators.

- Operators P_t^Λ are compact and selfadjoint,

Integrated density of states

Remedy: restrict the system to a finite volume, Λ.

- Only finite number of electrons in a finite volume.

- Laplacian in the finite volume: set boundary values (choose: Dirichlet or Neumann – which corresponds to killing or reflecting), then the semigroup $P_t^\Lambda = \mathrm{e}^{-t\Delta}$ consists of trace-class operators.

- Operators P_t^Λ are compact and selfadjoint, the generator of the semigroup has countably many eigenvalues,

Integrated density of states

Remedy: restrict the system to a finite volume, Λ.

- Only finite number of electrons in a finite volume.

- Laplacian in the finite volume: set boundary values (choose: Dirichlet or Neumann – which corresponds to killing or reflecting), then the semigroup $P_t^\Lambda = \mathrm{e}^{-t\Delta}$ consists of trace-class operators.

- Operators P_t^Λ are compact and selfadjoint, the generator of the semigroup has countably many eigenvalues, which are nonnegative, have no accumulation point other that $+\infty$,

$$0 \leqslant \lambda_0^\Lambda \leqslant \lambda_1^\Lambda \leqslant \ldots$$

Integrated density of states

Remedy: restrict the system to a finite volume, Λ.

- Only finite number of electrons in a finite volume.

- Laplacian in the finite volume: set boundary values (choose: Dirichlet or Neumann – which corresponds to killing or reflecting), then the semigroup $P_t^\Lambda = \mathrm{e}^{-t\Delta}$ consists of trace-class operators.

- Operators P_t^Λ are compact and selfadjoint, the generator of the semigroup has countably many eigenvalues, which are nonnegative, have no accumulation point other that $+\infty$,

$$0 \leqslant \lambda_0^\Lambda \leqslant \lambda_1^\Lambda \leqslant \ldots$$

- Object of interest:

the limiting behaviour of these spectra as $|\Lambda| \to \infty$.

Integrated density of states

Integrated density of states

Define:

$$\ell_\Lambda(\cdot) = \frac{1}{|\Lambda|} \sum_i \delta_{\{\lambda_i^\Lambda\}}(\cdot)$$

Integrated density of states

Define:

$$\ell_\Lambda(\cdot) = \frac{1}{|\Lambda|} \sum_i \delta_{\{\lambda_i^\Lambda\}}(\cdot)$$

Definition

The integrated density of states is, by definition, the vague limit of these measures as $|\Lambda| \to \infty$.

Integrated density of states

Define:

$$\ell_\Lambda(\cdot) = \frac{1}{|\Lambda|} \sum_i \delta_{\{\lambda_i^\Lambda\}}(\cdot)$$

Definition

The integrated density of states is, by definition, the vague limit of these measures as $|\Lambda| \to \infty$.

It can be understood as the 'number of energy levels per volume', when the volume is big.

Random interactions

Random interactions

We now alter the state-space

Random interactions

We now alter the state-space according to a Poisson point process on $\mathbb{R}^d$.

Random interactions

We now alter the state-space according to a Poisson point process on $\mathbb{R}^d$.

Let $\mathcal{N}(\omega)$ be the Poisson point process on $\mathbb{R}^d$, with intensity $\nu > 0$,

Random interactions

We now alter the state-space according to a Poisson point process on $\mathbb{R}^d$.

Let $\mathcal{N}(\omega)$ be the Poisson point process on $\mathbb{R}^d$, with intensity $\nu > 0$, defined on a probability space $(\Omega, \mathcal{M}, \mathbb{Q})$:

Random interactions

We now alter the state-space according to a Poisson point process on $\mathbb{R}^d$.

Let $\mathcal{N}(\omega)$ be the Poisson point process on $\mathbb{R}^d$, with intensity $\nu > 0$, defined on a probability space $(\Omega, \mathcal{M}, \mathbb{Q})$:

• For any Borel set $A \subset \mathbb{R}^d$ with $0 < |A| < \infty$, the number of Poisson points inside A, denoted by $\mathcal{N}(A)$, has Poisson distribution with parameter $\nu|A|$.

• When $A \cap B = \emptyset$, then $\mathcal{N}(A)$ and $\mathcal{N}(B)$ are independent random variables.

• Let $\mathcal{N}(\omega) = \{x_i\}$ denote the realization of the Poisson process.

• assume that the Poisson process and the Brownian motion are independent.

Poisson potential

Let $W \in C(\mathbb{R}^d) \geqslant 0$ with sufficiently fast decay at infinity, or $W \geqslant 0$, measurable and of compact support, $W(x) > a > 0$ on certain ball.
Then put

$$V(x,\omega) = \sum_i W(x - x_i) = \int_{\mathbb{R}^d} W(x-y)\mathrm{d}\mu^\omega(y),$$

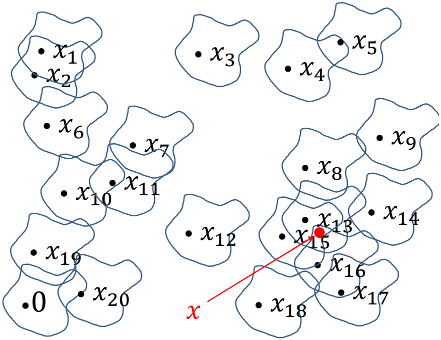
where μ^ω is the counting measure of a realization of the Poisson cloud.

Poisson point process and Poissonian random field

e.g. $W(x) := \mathbf{1}_E(x)$
for some $E \subset \mathcal{B}(\mathbb{R}^d)$

$$V^\omega(x) = \sum_{x_i \in \omega} \mathbf{1}_E(x - x_i)$$
$$= \sum_{x_i \in \omega} \mathbf{1}_{E+x_i}(x)$$

$$V^\omega(x) = \sum_{i \in \{13,15,16\}} \mathbf{1}_{E+x_i}(x)$$



Killing Poisson obstacles

- Fix $a > 0$, and remove [closed] balls with radius a, centered at the Poisson points, from the state-space.

- Denote the resulting set by $\mathcal{O}(\omega)$ and call it the free open set.

- Then consider the Brownian motion (X_t) (or another process on $\mathcal{O}(\omega)$): the Brownian motion is killed once it enters the obstacle set.

Random semigroup, its generator

- **Potential case**

The L^2-semigroup:

$$P_t f(x) = E_x[f(X_t)\mathrm{e}^{-\int_0^t V(X_s)\mathrm{d}x}],$$

generator: $Af = -\frac{1}{2}\Delta f + Vf$.

- **Killing obstacles case**

The L^2-semigroup:

$$P_t f(x) = E_x[f(X_t)\{\tau_{\mathcal{O}} > t\}]$$

generator: Laplacian with Dirichlet boundary values on the obstacle set.

- In either case the spectrum of the generator is not discrete.

• Consider the problems (either with killing obstacles, or with the potential) restricted to a big box $B(0, 2^M)$,
then the generator has a discrete spectrum,

$$0 \leqslant \lambda_1(M,\omega) \leqslant \lambda_2(M,\omega) \leqslant \ldots$$

• Consider the problems (either with killing obstacles, or with the potential) restricted to a big box $B(0, 2^M)$,
then the generator has a discrete spectrum,

$$0 \leqslant \lambda_1(M, \omega) \leqslant \lambda_2(M, \omega) \leqslant \ldots$$

• Set $\ell_M(\omega)(\cdot)$ to be the normalized counting measure of the spectrum of its generator (which is trace class):

$$\ell_M(\omega)(\cdot) = \frac{1}{|B(0, 2^M)|} \sum_i \delta_{\{\lambda_i(M,\omega)\}}(\cdot).$$

• Consider the problems (either with killing obstacles, or with the potential) restricted to a big box $B(0, 2^M)$,
then the generator has a discrete spectrum,

$$0 \leqslant \lambda_1(M,\omega) \leqslant \lambda_2(M,\omega) \leqslant \dots$$

• Set $\ell_M(\omega)(\cdot)$ to be the normalized counting measure of the spectrum of its generator (which is trace class):

$$\ell_M(\omega)(\cdot) = \frac{1}{|B(0,2^M)|} \sum_i \delta_{\{\lambda_i(M,\omega)\}}(\cdot).$$

• Then we have (classical result–Pastur/Figotin, Lifschitz...; ergodicity):

Theorem

$\mathbb{Q}$–*almost surely,*

• Consider the problems (either with killing obstacles, or with the potential) restricted to a big box $B(0, 2^M)$,
then the generator has a discrete spectrum,

$$0 \leqslant \lambda_1(M, \omega) \leqslant \lambda_2(M, \omega) \leqslant \ldots$$

• Set $\ell_M(\omega)(\cdot)$ to be the normalized counting measure of the spectrum of its generator (which is trace class):

$$\ell_M(\omega)(\cdot) = \frac{1}{|B(0, 2^M)|} \sum_i \delta_{\{\lambda_i(M,\omega)\}}(\cdot).$$

• Then we have (classical result–Pastur/Figotin, Lifschitz...; ergodicity):

Theorem

$\mathbb{Q}$–almost surely, the measures $\ell_M(\omega)$ converge vaguely to a nonrandom measure ℓ, concentrated on $[0, \infty)$ (the integrated density of states).

The Lifschitz singularity

The integrated density of states exhibits the so-called Lifschitz singularity at the origin:

$$\lim_{\lambda \searrow 0} \frac{\log I([0,\lambda])}{\lambda^{-d/2}} = C(d,\nu).$$

Generalizations

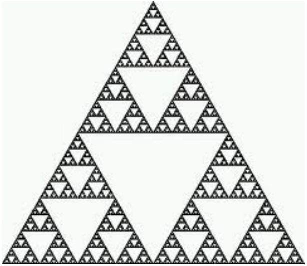
Want to have similar result for

- more general processes,
- more general state-space.

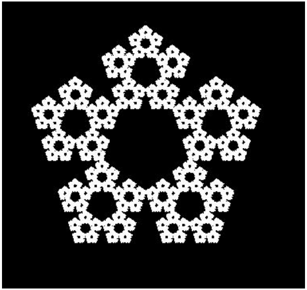
1. Lévy processes on $\mathbb{R}^d$, continuous potential (Nakao existence, Okura asymptotics),
2. Brownian motion on hyperbolic space, killing obstacles (Sznitman)
3. Brownian motion on the Sierpiński gasket, killing obstacles (P.-P.)

In (1) and (3) the convergence is very easy, in (2) it requires more work.

Nested fractals (embedded in $\mathbb{R}^n$)

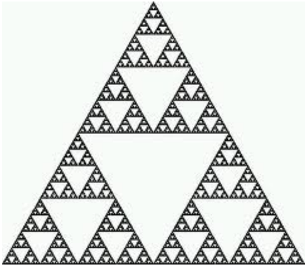


the Sierpiński gasket

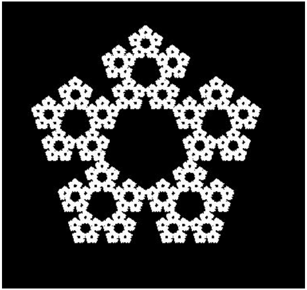


the snowflake

Nested fractals (embedded in $\mathbb{R}^n$)



the Sierpiński gasket



the snowflake

m—the Hausdorff measure on the fractal, in dimension d_f ($d_f = \frac{\log 3}{\log 2}$ on the gasket).

Markov processes on gaskets - Brownian motion

Brownian motion on the gasket (Barlow-Perkins 1989): strong Markov, Feller process Z_t with symmetric transition density that satisfies subgaussian estimated

$$c_1 t^{-\frac{d_f}{d_w}} e^{-c_2 \left(\frac{|x-y|}{t^{1/d_w}}\right)^{d_w/(d_w-1)}} \leqslant g(t,x,y) \leqslant c_3 t^{-\frac{d_f}{d_w}} e^{-c_4 \left(\frac{|x-y|}{t^{1/d_w}}\right)^{d_w/(d_w-1)}}$$

(d_w is the so-called walk dimension of the gasket, $d_w = \frac{\log 5}{\log 2}$).

Subordinate Brownian motion

Let $S = (S_t, \mathbf{P})_{t \geqslant 0}$ be a subordinator, i.e. an increasing Lévy process taking values in $[0, \infty]$ with $S_0 = 0$. The law of S will be denoted by $\eta_t(du)$.

We always assume that Z and S are independent. The process

$$X_t := Z_{S_t}, \quad t \geqslant 0,$$

is called the subordinate Brownian motion on the gasket $\mathcal{G}$ (via subordinator S). It is also a symmetric Markov process with transition probabilities given by

$$p(t, x, A) = \int_0^\infty \int_A g(u, x, y) m(dy) \eta_t(du), \\ t > 0, \quad x \in \mathcal{G}, \quad A \in \mathcal{B}(\mathcal{G}).$$

(need some additional assumptions on the subordinator)

Assumptions on the subordinator

1. Assumption 1.

$$\forall t > 0 \quad \int_0^\infty \frac{1}{u^{d_s/2}} \mathbf{P}[S_t \in \mathrm{d}u] =: c_0(t) < \infty,$$

2. Assumption 2.

$$\int_0^\infty \ln(u \vee 1) \mathbf{P}[S_t \in \mathrm{d}u] = \mathbf{E}[\ln(S_t \vee 1)] < \infty.$$

Under assumption 1, the process has symmetric, strictly positive transition densities given by

$$p(t,x,y) = \int_0^\infty g(u,x,y) \mathbf{P}[S_t \in \mathrm{d}u]$$

that inherits all regularity properties of g.

Potentials

- Random potentials:

$$V(x,\omega) := \int_{\mathcal{G}} W(x,y)\mu^{\omega}(dy),$$

where μ^{ω} is the random counting measure corresponding to the Poisson random measure on $\mathcal{G}$, with intensity $\nu \mathrm{d}m$, $\nu > 0$, defined on a probability space $(\Omega, \mathcal{M}, \mathbb{Q}))$, and $W : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is a measurable, nonnegative profile function.

- We assume that the Poisson process and the Markov process X are independent.

Kato class

- A Borel function V is in Kato class $\mathcal{K}^X$ related to the process X_t if

$$\lim_{t \searrow 0} \sup_{x \in \mathcal{G}} \int_0^t \mathbf{E}_x |V(X_s)| ds = 0.$$

- $V \in \mathcal{K}^X_{loc}$ (local Kato class), when $\mathbf{1}_B V \in \mathcal{K}^X$ for every ball $B \subset \mathcal{G}$.

Examples of potentials

Example 1. $W(x,y) = \phi(d(x,y))$, with $\phi : [0,\infty) \to [0,\infty)$ of compact support, separated from 0 in the vicinity of 0, $\phi(d(\cdot,y)) \in \mathcal{K}^X_{loc}$ for every $y \in \mathcal{G}$.

Example 2. Fix $M \in \mathbb{Z}_+$. Set $W(x,y) = 1$ if x and y belong to the same gasket triangle of level M, and 0 otherwise.

Examples of potentials

Example 1. $W(x,y) = \phi(d(x,y))$, with $\phi : [0,\infty) \to [0,\infty)$ of compact support, separated from 0 in the vicinity of 0, $\phi(d(\cdot,y)) \in \mathcal{K}^X_{loc}$ for every $y \in \mathcal{G}$.

Example 2. Fix $M \in \mathbb{Z}_+$. Set $W(x,y) = 1$ if x and y belong to the same gasket triangle of level M, and 0 otherwise.

Remark. Profiles from Examples 1,2 have finite range. Certain profiles of infinite range are permitted as well.

The Dirichlet semigroup

When $\mathbb{Q}-$a.s. $V(\cdot,\omega)$ is in the local Kato class, then for $t > 0$, $M \in \mathbb{Z}_+$

$$P_t^{D,M,\omega} f(x) = \mathbf{E}_x \left[e^{-\int_0^t V(X_s) ds} f(X_t); t < \tau_{\mathcal{G}_M} \right], \quad f \in L^2(\mathcal{G}_M, m),$$

define ($\mathbf{Q}-$almost surely) a trace-class semigroup of operators; its generator have a discrete spectrum . This corresponds to *Dirichlet boundary conditions* outside $\mathcal{G}_M$, where $\mathcal{G}_M$ is the blowup of the unit gasket $\mathcal{G}_0$ by the factor 2^M.

- Denote the eigenvalues of the generator by

$$0 \leqslant \lambda_1^{D,M}(\omega) \leqslant \lambda_2^{D,M}(\omega) \leqslant \dots$$

- Goal: to establish the vague convergence of the measures

$$l_M^D(\omega) := \frac{1}{m(\mathcal{G}_M)} \sum_{n=1}^{\infty} \delta_{\lambda_n^{D,M}(\omega)}.$$

How to do this?

- The Laplace transform of the measure $l_M^D(\omega)$ is

$$\begin{aligned} L_M^D(t,\omega) &= \int_0^\infty e^{-\lambda t} \mathrm{d} l_M^D(\omega)(t) \\ &= \frac{1}{m(\mathcal{G}_M)} \sum_{n=1}^{\infty} e^{-\lambda_n^{D,M}(\omega) t} = \frac{1}{m(\mathcal{G}_M)} \mathrm{Tr}\; T_t^{D,M}. \end{aligned}$$

Ingredients of the proof:

- the convergence of $\mathbb{E}_{\mathbb{Q}} L_M^D(t,\omega)$ as $M \to \infty$,
- the convergence of the series

$$\sum_M \mathrm{Var}\, L_M^D(t,\omega) < \infty.$$

This would be enough: Borel-Cantelli lemma argument + properties of vague convergence expressed as the convergence of Laplace transforms give the result.

The reflected process

Previously, the convergence of the expected values was easy, the convergence of the series of variances was more difficult.

Problems:

(1) No translation invariance – ergodic methods not applicable.

(2) This problem was present also for the Brownian motion, but the Brownian motion on the gasket has lots of symmetries, which helps. Here the symmetries are destroyed.

We want to recover some of the symmetries. To this end, we introduce the "reflected subordinate Brownian motion on $\mathcal{G}_M$,", which would correspond to taking Neumann boundary conditions in the diffusion case.
Denote X_t^M–the reflected subordinate Brownian motion on $\mathcal{G}_M$.

The Neumann semigroup

The semigroup

$$P_t^{N,M,\omega} f(x) = \mathbf{E}_x^M \left[e^{-\int_0^t V(X_s^M) ds} f(X_t^M) \right], \quad f \in L^2(\mathcal{G}_M, m),$$

is also trace-class, and its generator has a complete set of eigenfunctions. Denote them

$$0 \leqslant \lambda_1^{N,M}(\omega) \leqslant \lambda_2^{N,M}(\omega) \leqslant \dots$$

and consider

$$l_M^N(\omega) := \frac{1}{m(\mathcal{G}_M)} \sum_{n=1}^{\infty} \delta_{\lambda_n^{N,M}(\omega)}.$$

Proposition

Let $0 \leqslant V(\cdot,\omega) \in \mathcal{K}_{loc}^X$ *for* $\mathbb{Q}$*-almost all* ω*. Then for every* $t > 0$*,* $L_M^N(t,\omega)$ *and* $L_M^D(t,\omega)$ *satisfy*

$$\sum_{M=1}^{\infty} \mathbb{E}_{\mathbb{Q}} \left(L_M^N(t,\omega) - L_M^D(t,\omega) \right)^2 < \infty.$$

Therefore it would be enough to get the results for the Neumann boundary conditions.

Still, we are not able to prove that $\mathbb{E}_{\mathbb{Q}} L_M^N(t,\omega)$ is convergent.

Need more assumptions concerning the profile function W.

1. $\forall y \in \mathcal{G}$ one has $W(\cdot, y) \in \mathcal{K}^X_{loc}$.
2. $$\sup_{x \in \mathcal{G}} W(x,y) \leqslant h(y) \in L^1(\mathcal{G}, m).$$
3. $$\sum_{y' \in \pi_M^{-1}(\pi_M(y))} W(\pi_M(x), y') = \sum_{y' \in \pi_M^{-1}(\pi_M(y))} W(x, y'), \quad x, y \in \mathcal{G}, \tag{1}$$

   for all sufficiently large M.

Under this assumption, we introduce the following ‘periodization’ of the random potential V.

Definition

The family of random fields $(V_M^*)_{M\in\mathbb{Z}_+}$ on $\mathcal{G}$ given by

$$V_M^*(x,\omega) := \int_{\mathcal{G}_M} \sum_{y'\in\pi_M^{-1}(y)} W(x,y')\mu^\omega(dy), \quad M\in\mathbb{Z}_+$$

is called the M-periodization of V *in the Sznitman sense*.

Now we use measures $I_M^{N^*,\omega}$ – corresponding to the potential V_M^*. and we consider the Laplace transforms of these measures, $L_M^{N^*}(t,\omega)$.

The results

Theorem (K.Kaleta, K.P.-P.)

(1) For any $t > 0$ there exists a finite number $L(t)$ such that, under some regularity assumptions on the profile function W and the subordinator S_t,

$$\lim_{M\to\infty} \mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t,\omega) = L(t),$$

(2)

$$\mathbb{E}_{\mathbb{Q}}(L_M^{N^*}(t,\omega) - L_M^D(t,\omega)) = o(1), \quad M \to \infty,$$

(3)

$$\sum_{M=1}^{\infty} \mathbb{E}_{\mathbb{Q}}[L_M^D(t,\omega) - \mathbb{E}_{\mathbb{Q}} L_M^D(t,\omega)]^2 < \infty \qquad (2)$$

Results (2) and (3) are true for $L_M^N(t,\omega)$ as well.

Proof of (1):

Observe that once the path of the process X_t is fixed, then we have

$$\mathbb{E}_{\mathbb{Q}} e_{(V_M^*)_M}(t) = \mathbb{E}_{\mathbb{Q}} e_{V_M^*}(t), \quad t > 0, \tag{3}$$

and that the monotonicity holds

$$\mathbb{E}_{\mathbb{Q}} e_{V_{M+1}^*}(t) \leqslant \mathbb{E}_{\mathbb{Q}} e_{V_M^*}(t), \quad t > 0, \tag{4}$$

where $e_V(t) = \mathrm{e}^{-\int_0^t V(X_s)\mathrm{d}s}$. This results in the monotonicity of $\mathbb{E}_{\mathbb{Q}} L_M^{N^*}(t, \omega)$, which is in this case nonincreasing.

Conclusion

Corollary

$\mathbb{Q}-$a.s., the measures $I_M^D(\omega)$ and $I_M^N(\omega)$ converge to a common nonrandom limit I, which is a measure on $\mathbb{R}_+$. This limit is called the integrated density of states.

The asymptotics of the Laplace transform. Stable case

The asymptotics of the Laplace transform. Stable case

Once the integrated density of states is well defined,

The asymptotics of the Laplace transform. Stable case

Once the integrated density of states is well defined, we can investigate its asymptotics.

The asymptotics of the Laplace transform. Stable case

Once the integrated density of states is well defined, we can investigate its asymptotics.

We work with the Laplace transform of the IDS, $L(t)$.

The asymptotics of the Laplace transform. Stable case

Once the integrated density of states is well defined, we can investigate its asymptotics.

We work with the Laplace transform of the IDS, $L(t)$.

Case 1. Stable processes on fractals, killing obstacles

Theorem (D. Kowalska and K.P.-P.)

Suppose $\nu > 0$ is the intensity of the Poisson point process on the gasket. Let $L(t)$ be the Laplace transform of the integrated density of states for the $\alpha-$stable process on the Sierpiński gasket and killing Poissonian obstacles. Then there exist two positive constants C_1, C_2 such that:

$$-C_1 \nu^{\frac{\alpha}{2} d_w / d_\alpha} \leqslant \liminf_{t\to\infty} \frac{\log L(t)}{t^{d_f/d_\alpha}} \leqslant \limsup_{t\to\infty} \frac{\log L(t)}{t^{d_f/d_\alpha}} \leqslant -C_2 \nu^{\frac{\alpha}{2} d_w / d_\alpha}$$

The asymptotics of the Laplace transform. Stable case

Once the integrated density of states is well defined, we can investigate its asymptotics.

We work with the Laplace transform of the IDS, $L(t)$.

Case 1. Stable processes on fractals, killing obstacles

Theorem (D. Kowalska and K.P.-P.)

Suppose $\nu > 0$ is the intensity of the Poisson point process on the gasket. Let $L(t)$ be the Laplace transform of the integrated density of states for the $\alpha-$stable process on the Sierpiński gasket and killing Poissonian obstacles. Then there exist two positive constants C_1, C_2 such that:

$$-C_1 \nu^{\frac{\alpha}{2} d_w / d_\alpha} \leqslant \liminf_{t \to \infty} \frac{\log L(t)}{t^{d_f / d_\alpha}} \leqslant \limsup_{t \to \infty} \frac{\log L(t)}{t^{d_f / d_\alpha}} \leqslant -C_2 \nu^{\frac{\alpha}{2} d_w / d_\alpha}$$

where $d_\alpha = d_f + \frac{\alpha}{2} \frac{\log 5}{\log 2}$.

The asymptotics of the IDS

The asymptotics of the IDS

It is classical to transform the asymptotics of $L(t)$ as $t \to \infty$ to the asymptotics of $\ell(0, \lambda]$ as $\lambda \to 0^+$.

The asymptotics of the IDS

It is classical to transform the asymptotics of $L(t)$ as $t \to \infty$ to the asymptotics of $\ell(0, \lambda]$ as $\lambda \to 0^+$. This uses Tauberian theorems.

The asymptotics of the IDS

It is classical to transform the asymptotics of $L(t)$ as $t \to \infty$ to the asymptotics of $\ell(0,\lambda]$ as $\lambda \to 0^+$. This uses Tauberian theorems. We get:

Theorem

There exist two constants: $C > 0$ and $D > 0$ such that

$$-C\nu \leqslant \liminf_{\lambda\to 0} \lambda^{d_s/\alpha} \log \ell([0,\lambda]) \leqslant \limsup_{\lambda\to 0} \lambda^{d_s/\alpha} \log \ell([0,\lambda]) \leqslant -D\nu.$$

The asymptotics of the Laplace transform. General case.

The asymptotics of the Laplace transform. General case.

Suppose L is a generator of a subordinate Brownian motion whose characteristic exponent $\phi(\lambda) = b\lambda + \psi(\lambda)$ satisfies:

(L1) There exist constants $c_{3.1} > 0$, $\beta \in (0, d_w]$ and $s_0 > 0$ such that for $s \in (0, s_0]$ one has $\phi(s) \leqslant c_{3.1} s^{\beta/d_w}$.

(U1) $b > 0$ and $\psi \equiv 0$ (equivalently, $\nu \equiv 0$; no jumps)

or

(U2) $b > 0$ and $\psi \neq 0$ satisfies the following weak scaling conditions: there are $\alpha_1, \alpha_2, \beta, \delta \in (0, d_w)$, $a_1, a_2 \in (0, 1]$, $a_3, a_4 \in [1, \infty)$ and $r_0 > 0$ such that

$$a_1 \lambda^{\alpha_1/d_w} \psi(r) \leqslant \psi(\lambda r) \leqslant a_3 \lambda^{\beta/d_w} \psi(r), \ \lambda \in (0, 1], \ r \leqslant r_0$$

$$a_2 \lambda^{\alpha_2/d_w} \psi(r) \leqslant \psi(\lambda r) \leqslant a_4 \lambda^{\delta/d_w} \psi(r), \ \lambda \geqslant 1, \ r \geqslant r_0$$

or

(U3) $b = 0$ and $\psi \neq 0$ satisfies the above with $\alpha_1 = \alpha_2$.

Examples of subordinators and subordinate processes

- *Pure drift.* Let $\phi(\lambda) = b\lambda$, $b > 0$. The corresponding subordinate process is just the Brownian motion with speed $b > 0$.
- *Stable subordinators.* Let $\phi(\lambda) = \lambda^{\gamma/d_w}$, $\gamma \in (0, d_w)$.
- *Stable subordinators with drift.* Let $\phi(\lambda) = b\lambda + \lambda^{\gamma/d_w}$, $\gamma \in (0, d_w)$, $b > 0$. Then the corresponding subordinator is a sum of a pure drift subordinator bt and the pure jump γ/d_w-stable subordinator.
- *Mixture of purely jump stable subordinators.* Let $\phi(\lambda) = \sum_{i=1}^{n} \lambda^{\gamma_i/d_w}$, $\gamma_i \in (0, d_w)$, $n \in \mathbb{N}$.
- Let $\phi(\lambda) = b\lambda + \lambda^{\gamma_1/d_w}[\log(1+\lambda)]^{\gamma_2/d_w}$, $\gamma_1 \in (0, d_w)$, $\gamma_2 \in (-\gamma_1, d_w - \gamma_1)$, $b > 0$.

Let $\gamma = d_w$ under **(U1)**, and $\gamma = \alpha_1$ under **(U2)** or **(U3)**.

Theorem (K. Kaleta, KPP)

Suppose X is a subordinate Brownian motion in $\mathcal{G}$ via a complete subordinator S with Laplace exponent ϕ satisfying **(L1)** *and* **(U1)**, **(U2)**, *or* **(U3)**. *Let the profile W be regular enough and of compact support. Then there exist constants $C_1, C_2 > 0$ such that for every $x \in \mathcal{G}$:*

$$\limsup_{t\to\infty} \frac{\log L(t)}{t^{\frac{d_f}{d_f+\gamma}}} \leqslant -C_1 \nu^{\frac{\gamma}{d_f+\gamma}},$$

$$\liminf_{t\to\infty} \frac{\log L(t)}{t^{\frac{d_f}{d_f+\beta}}} \geqslant -C_2 \nu^{\frac{\beta}{d_f+\beta}}.$$

Remark

Tauberian theorems can be used to transform these into bounds for $I(\lambda)$ near zero. If $\gamma = \beta$ then we recover the asymptotics we had for killing obstacles and stable processes:

$$\liminf_{x \to 0} x^{d_f/\beta} \log I([0,x]) \geqslant -C\nu,$$

$$\limsup_{x \to 0} x^{d_f/\gamma} \log I([0,x]) \leqslant -C\nu,$$

(for stable processes $\beta = \gamma = \alpha d_w$).

More general potentials

Assume that

(WW) there exist $\theta > 0$, $K_1, K_2 \geqslant 0$ such that:

$$\begin{aligned} K_1 &= \liminf_{d(x,y)\to\infty} W(x,y)d(x,y)^{d_f+\theta} \\ &\leqslant \limsup_{d(x,y)\to\infty} W(x,y)d(x,y)^{d_f+\theta} = K_2 < \infty. \end{aligned}$$

Lower bound, the general case.

Theorem

Under same assumptions plus **(WW)**, *thefe exist constants C_1, C_1 such that:*
(i) when $\beta < \theta$ then

$$\liminf_{t\to\infty} \frac{\log L(t)}{t^{d/(d+\beta)}} \geqslant -C_1 \nu^{\beta/(d+\beta)},$$

(ii) when $\beta = \theta$ then

$$\liminf_{t\to\infty} \frac{\log L(t)}{t^{d/(d+\beta)}} \geqslant -C_1 \nu^{\beta/(d+\beta)} - C_1' \nu,$$

(iii) when $\beta > \theta$ then

$$\liminf_{t\to\infty} \frac{\log L(t)}{t^{d/(d+\theta)}} \geqslant -C_1' \nu.$$

Matching upper bound, the general case.

Theorem

Under same assumptions as above plus **(WW)**, *there exist constants* $E_1, E_1' > 0$ *such that:*
(i) when $\gamma < \theta$ *then*

$$\limsup_{t\to\infty} \frac{\log L(t)}{t^{d/(d+\gamma)}} \leqslant -E_1 \nu^{\gamma/(d+\gamma)},$$

(ii) when $\gamma = \theta$ *then*

$$\limsup_{t\to\infty} \frac{\log L(t)}{t^{d/(d+\gamma)}} \leqslant -E_1 \nu^{\gamma/(d+\gamma)} - E_1' \nu,$$

(iii) when $\gamma > \theta$ *then*

$$\limsup_{t\to\infty} \frac{\log L(t)}{t^{d/(d+\theta)}} \leqslant -E_1' \nu.$$

What's next?

More general fractals!

Work in progress, joint with **Kamil Kaleta and Michał Olszewski**.

Thank you for your attention!