Stability inequalities for martingales and Riesz transforms*

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Stability (quantitative/deficit) sharp inequalities

Optimal/sharp inequalities

Suppose you have two functionals $\mathcal{E}$ and $\mathcal{F}$ on some normed (real) linear space $\mathcal{M}$ satisfying the functional inequality $\mathcal{E} \leqslant \mathcal{F}$ in the sense that

$$\mathcal{E}(x) \leqslant \mathcal{F}(x), \quad \forall x \in \mathcal{M}.$$

The functional inequality $\mathcal{E} \leqslant \mathcal{F}$ is sharp if for all $\lambda < 1$ there exist $x \in \mathcal{M}$ such that

$$\mathcal{E}(x) > \lambda \mathcal{F}(x)$$

The subset

$$\mathcal{M}_0 = \{x \in \mathcal{M} : \mathcal{E}(x) = \mathcal{F}(x)\}$$

is called the set of optimizers (extremals) of the inequality. When $\mathcal{M}_0 \neq \emptyset$, the inequality is said to be optimal. (Note: An optimal functional inequality is sharp but not vice–versa.)

Definition

Let d be a metric on $\mathcal{M}$ (not necessarily the norm metric) and Φ a "rate function." The optimal functional inequality $\mathcal{E} \leqslant \mathcal{F}$ is (d, Φ)– stable if

$$\mathcal{F}(x) - \mathcal{E}(x) \geqslant \Phi(d(x, \mathcal{M}_0)), \quad \forall x \in \mathcal{M}$$

In various examples, $\Phi(t) = ct^2$ and $d(x,y) = \|x-y\|_{\mathcal{M}}$ and

$$\mathcal{F}(x) - \mathcal{E}(x) \geqslant c \inf_{z \in \mathcal{M}_0} \|x - z\|^2_{\mathcal{M}}.$$

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- Classical Sobolev in $\mathbb{R}^n$ $(n \geqslant 3)$. $\quad k_n^2 = \frac{n(n-2)}{4}|\mathbb{S}^{n-1}|$

$$k_n^2 \|f\|_{\frac{2n}{n-2}}^2 \leqslant \|\nabla f\|_2^2, \quad \forall f \in H_0^1(\mathbb{R}^n) = \mathcal{M},$$

$$\mathcal{M}_0 = \{x \to c(a + b|x - x_0|^2)^{-(n-2)/2}, a, b > 0, x_0 \in \mathbb{R}^n, c \in \mathbb{R}\}$$

Optimality: Aubin (1976), Talenti (1976). Stability: Biachi-Egnell (1990)

$$\|\nabla f\|_2^2 - k_n^2 \|f\|_{\frac{2n}{n-2}}^2 \geqslant C \inf_{g \in \mathcal{M}_0} \|\nabla(f - g)\|_2^2$$

- More general Sobolev ($0 < \alpha < n/2$).

$$\|f\|_{\frac{2n}{n-2\alpha}} \leqslant k_{n,\alpha} \|(-\Delta)^{\alpha/2} f\|_2$$

Optimality: Lieb (1983), Stability: Cheng-Frank-Weth (2013)

- Hardy-Littlewood-Sobolev Optimality: Lieb (1983). Stability: Carlen (2016), Log-Sobolev Gross (1975), Stability Fathi-Indrei-Ledoux (2015), Nash Optimality: Carlen-Loss (1993). Stability: Carlen-Lieb (2017)

- Housdorff-Young inequality: Sharpness Beckner 1975 (Lieb 1990) Stability: Chris 2015. $1 \leqslant p \leqslant 2$, $q = \frac{p}{p-1}$

$$\|\hat{f}\|_q \leqslant (A_p)^n \|f\|_p \quad A_p = p^{1/2p} q^{-1/2q}$$

A_p is best contacts. Extremizers are general Gaussians: $g(x) = ce^{Q(x)+x\cdot v}$.

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- Brasco &De Philippis (2016). Torsional rigidity for Brownian motion.

$$\int_{D*} \mathbb{E}_z(\tau_{D^*})dz - \int_D \mathbb{E}_z(\tau_D)dz \geqslant C_n \mathcal{A}(D)^2$$

(*Fraenkel Asymmetry*) $\mathcal{A}(D) := \inf\{\frac{|D \triangle B|}{|D|} : B$ is a ball with $|B| = |D|\}$.

Problem: Prove "it" for stable processes (any subordination of BM).

Martingales: Sharp (but not optimal, i.e., $\mathcal{M}_0 = \emptyset$)) inequalities.

(Reference: A. Osękowski, *"Sharp martingale and semimartingale inequalities"*, Monografie Matematyczne **72**, Birkhäuser, 2012.)

Doob

$\{f_n\}$ an L^p, $1 < p \leqslant \infty$ martingale. $f^* = \sup_n |f_n|$ maximal function.

$$\|f^*\|_p \leqslant \frac{p}{p-1}\|f\|_p$$

Burkholder (1984), Wang (1991 for dyadic): Inequality is sharp. But $\mathcal{M}_0 = \emptyset$.

Burkholder (1966) $S(f) = (\sum_n (f_n - f_{n-1})^2)^{1/2}$

$$a_p\|f\|_p \leqslant \|S(f)\|_p \leqslant b_p\|f\|_p \quad 1 < p < \infty$$

Many sharp versions of these exists but none are optimal i.e., $\mathcal{M}_0 = \emptyset$, outside of the trivial case of $p = 2$. (The first sharp case of these for Brownian martingales/stochastic integrals is due to B. Davis (1976).)

X, Y cádlág (right continuous/left limits) martingales:

- Y is differentially subordinate to X ($Y << X$), if the process $\{[X,X]_t - [Y,Y]_t\}_{t \geqslant 0}$ is nonnegative and nondecreasing in t.
- They are orthogonal ($Y \perp X$) if $[X,Y] = 0$. $1 < p < \infty$ and .

$p^* = \max\{p, p/(p-1)\}$. Set $||X||_p = \sup_{t \geqslant 0} ||X_t||_p$

Burkholder (1984) $Y << X$

$$||Y||_p \leqslant (p^* - 1)||X||_p.$$

The constant $(p^* - 1)$ is best possible. Furthermore, the inequality is always strict unless $p = 2$. That is, inequality is sharp and $\mathcal{M}_0 = \emptyset$ unless $p = 2$.

R.B. G. Wang (1995) $Y << X$ and $Y \perp X$

$$||Y||_p \leqslant cot\left(\frac{\pi}{2p^*}\right)||X||_p.$$

The constant $cot\left(\frac{\pi}{2p^*}\right)$ is the best possible. Furthermore, the inequality is always strict unless $p = 2$ unless $p = 2$. That is, inequality is sharp and $\mathcal{M}_0 = \emptyset$ unless $p = 2$.

A careful analysis of proofs reveals that "almost" extremals used to proof sharpness are "almost" eigenfunctions.

The dyadic maximal function in $\mathbb{R}^n$ (dyadic martingales).

$$M_d(f)(x) = \sup\{\frac{1}{|Q|}\int_Q |f(y)|dy : x \in Q, Q \in [0,1]^n, \text{dyadic cube}\}$$

Doob (for inequality) Wang (1995) for sharpness: $\|M_d(f)\|_p \leqslant \frac{p}{p-1}\|f\|_p$, $1 < p \leqslant \infty$. (Here we may restrict to non-negative functions.)

A. Melas (2015): If you take a sequence $\{f_n\}$ of almost externals then $\lim_n \|M_d(f_n) - \frac{p}{p-1}f_n\|_p = 0$

Theorem (Melas 2015)

Fix $2 < p < \infty$, $\epsilon > 0$ (small enough). Suppose $f \geqslant 0$ (in L^p) is such that

$$\|M_d(f)\|_p \geqslant \left(\frac{p}{p-1} - \varepsilon\right)\|f\|_p.$$

Then

$$\|M_d(f) - \frac{p}{p-1}f\|_p \leqslant c_p \varepsilon^{1/p}\|f\|_p$$

for some constant c_p depending only on p.

Theorem (A.Osękowski & R.B. 2016: $Y << X$)

(i) Let $1 < p < 2$ *and* $\varepsilon > 0$. $||Y||_p \geqslant (\frac{1}{p-1} - \varepsilon)||X||_p$. *Then*

$$\Big|\Big||Y_\infty| - \frac{1}{(p-1)}|X_\infty|\Big|\Big|_p \leqslant c_p \varepsilon^{1/2}||X||_p.$$

$O(\varepsilon^{1/2})$ *as* $\varepsilon \to 0$ *is sharp.* $c_p = O((2-p)^{-1/2})$ *as* $p \uparrow 2$ *and this is sharp.*

(ii) Let $2 < p < \infty$ *and* $\varepsilon > 0$. $||Y||_p \geqslant (p - 1 - \varepsilon)||X||_p$.

$$\Big|\Big||Y_\infty| - (p-1)|X_\infty|\Big|\Big|_p \leqslant c_p \varepsilon^{1/p}||X||_p,$$

$O(\varepsilon^{1/p})$ *as* $\varepsilon \to 0$ *is sharp.* c_p *is* $O((p-2)^{-1/p})$ *as* $p \downarrow 2$ *and* $O(p)$ *as* $p \to \infty$. *These orders are sharp.*

(iii) For $p = 2$, *no* c_2 *and* κ *exist such that* $||Y||_2 \geqslant (1-\varepsilon)||X||_2$ *implies* $\Big|\Big||Y_\infty| - |X_\infty|\Big|\Big|_2 \leqslant c_2 \varepsilon^{\kappa}||X||_2$. *In fact, there exist martingales* Y *and* X, $Y << X$, *such that*

$$\|Y\|_2 = \|X\|_2, \quad \textit{and} \quad \frac{\||Y| - |X|\|_2}{\|X\|_2} > 0 \ \ (\textit{independent of } \varepsilon)$$

Theorem (A.Osękowski & R.B. 2016: $Y << X$ and $Y \perp X$)

(i)

$$\left\| |Y_\infty| - tan\left(\frac{\pi}{2p}\right)|X_\infty| \right\|_p \leqslant c_p \varepsilon^{1/2} ||X||_p, \;\; 1 < p < 2,$$

if X and Y are such that

$$||Y||_p \geqslant \left(tan\left(\frac{\pi}{2p}\right) - \varepsilon\right)||X||_p.$$

Orders in ε as $\varepsilon \to 0$, and c_p, as $p \uparrow 2$, are best possible.

(ii)

$$\left\| |Y_\infty| - cot\left(\frac{\pi}{2p}\right)|X_\infty| \right\|_p \leqslant c_p \varepsilon^{1/p} ||X||_p, \;\; 2 < p < \infty,$$

if

$$||Y||_p \geqslant \left(cot\frac{\pi}{2p} - \varepsilon\right)||X||_p$$

(iii) As in previous theorem, no such estimate exists for $p = 2$.

$$Bf(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,\mathrm{d}w.$$

It is a Calderón-Zygmund singular integral and $\|Bf\|_p \leqslant C_p\|f\|_p$, $1 < p < \infty$.

Conjecture T. Iwaniec 1984

The operator norm of B on L^p is $p^* - 1$:

$$\|B\|_{p\to p} = (p^* - 1), \;\; 1 < p < \infty$$

Known:

$$(p^* - 1) \leqslant \|B\|_{p\to p} \leqslant 1.575(p^* - 1)$$

Lower bound O. Lehto (1965) upper bound R.B and P. Janakiraman (2008).

Lehto's functions used to prove the lower bound have the property that $|Bf(z)| \approx (p^* - 1)|f(z)|$. That is, they are "near eigenfunctions."

$$\widehat{Bf}(\xi) = \frac{\overline{\xi}}{\xi}\widehat{f}(\xi) = \frac{\overline{\xi}^2}{|\xi|^2}\hat{f}(\xi) = \frac{\xi_1^2 - 2i\xi_1\xi_2 - \xi_2^2}{|\xi|^2}\hat{f}(\xi)$$

$$\Rightarrow \quad B = R_1^2 - R_2^2 + 2iR_1R_2 = \mathrm{Re}(B) + i\,\mathrm{Im}(B)$$

where R_1 and R_2 are the Riesz transforms in $\mathbb{R}^2$.

1. R. B. Wang (1995): Both Re(B) and $\mathrm{Im}(B)$ have norm$\leqslant (p^*-1)$ (Proving $\|B\|_p \leqslant 4(p^*-1)$)
2. Nazarov and Volberg (2004):

$$\|R_j^2 - R_k^2\|_{p\to p} \leqslant (p^*-1)$$

$$\|2R_jR_k\|_{p\to p} \leqslant (p^*-1)$$

   (Proving $\|B\|_{p,p} \leqslant 2(p^*-1)$)

3. Geiss, Montgomery-Smith and Saksman (2009):

$$\|R_j^2 - R_k^2\|_{p\to p} = (p^*-1), \quad \|2R_jR_k\|_{p\to p} = (p^*-1), j \neq k$$

Theorem (A.Osękowski & R.B. 2016)

T either $\mathrm{Re}(B)$ or $\mathrm{Im}(B)$ or more generally, $R_j^2 - R_k^2$ or $2R_jR_k$, $j \neq k$ in $\mathbb{R}^n$.
(i) Let $1 < p < 2$, $\varepsilon > 0$. If $f \in L^p(\mathbb{R}^n)$ is such that

$$||Tf||_p \geqslant ((p-1)^{-1} - \varepsilon)||f||_p,$$

then

$$|||Tf| - (p-1)^{-1}|f|||_p \leqslant c_p \varepsilon^{1/2}||f||_p.$$

The order $O(\varepsilon^{1/2})$, as $\varepsilon \to 0$, is best possible as is the order $O((2-p)^{-1/2})$, as $p \uparrow 2$ for c_p. (c_p is explicit and independent of dimension)
(ii) Let $2 < p < \infty$, $\varepsilon > 0$. If $f \in L^p(\mathbb{R}^n)$ is such that

$$||Tf||_p \geqslant (p - 1 - \varepsilon)||f||_p,$$

then

$$|||Tf| - (p-1)|f|||_p \leqslant c_p \varepsilon^{1/p}\|f\|_p,$$

(iii) For $p = 2$, there is no stability result of the above type. That is, there are no finite constants c_2 and $\kappa > 0$ such that

$$|||Tf| - |f|||_p \leqslant c_2 \varepsilon^{\kappa}||f||_{L^2(\mathbb{R}^d)}$$

Calderón-Zygmund singular integrals and also Fourier multipliers with

$$\widehat{R_j f}(\xi) = \frac{-i\xi_j}{|\xi|}, \qquad j = 1, 2, \ldots, n.$$

For $n = 1$ this is just the Hilbert transform.

$n = 1$: Pichorides (1972). $n > 1$: Iwaniec-Martin (1995)

$$||R_j f||_{L^p(p} \leqslant cot \frac{\pi}{2p^*} ||f||_p, \qquad j = 1, 2, \ldots, n$$

and this is sharp.

R.B. Wang (1995) obtained it from Orthogonal martingales.

Theorem (A.Osękowski & R.B. 2016)

Same type of stability results hold for R_j. (i) Let $1 < p < 2$, $\varepsilon > 0$.

$$||R_j f||_p \geqslant \left(tan\left(\frac{\pi}{2p}\right) - \varepsilon\right) ||f||_p \Rightarrow \||R_j f| - tan\left(\frac{\pi}{2p}\right)|f|\|_p \leqslant c_p \varepsilon^{1/2} ||f||_p,$$

$$f_n = \sum_{k=1}^{n} d_k, \quad g = \sum_{k=1}^{n} e_k, \quad |e_k| \leqslant |d_k|, \ a.s. \ \forall k$$

Burkholder's inequality: considers the function $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$V_p(x, y) = |y|^p - (p^* - 1)^p |x|^p.$$

The goal is then to show that $EV(f_n, g_n) \leqslant 0$. Burkholder then "introduces" the function

$$U_p(x, y) = \beta_p \left(|y| - (p^* - 1)|x|\right) \left(|x| + |y|\right)^{p-1},$$

where

$$\beta_p = p \left(1 - \frac{1}{p^*}\right)^{p-1}$$

and proves that this function satisfies the following properties:

$$V_p(x, y) \leqslant U(x, y) \text{ for all } x, y \in \mathbb{R}$$

and

$$EU_p(f_n, g_n) \leqslant EU(f_{n-1}, g_{n-1}) \leqslant \cdots \leqslant EU(f_0, g_0) = 0$$

Lemma (G. Wang (1995))

Let $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is "smooth" (outside the origin with certain bounds, $\ldots$) and satisfying (for all $h,\ k \in \mathbb{R}$)

$$U_{xx}(x,y)|h|^2 + 2U_{xy}(x,y)hk + U_{yy}(x,y)|k|^2 \leqslant -c_i(x,y)(|h|^2 - |k|^2).$$

Then if X, Y are two martingales such that $Y << X$, there is a nondecreasing sequence $(\tau_n)_{n\geqslant 1}$ of stopping times converging to infinity such that

$$\mathbb{E}U(X_{\tau_n\wedge t}, Y_{\tau_n\wedge t}) \leqslant \mathbb{E}U(X_0, Y_0), \qquad n = 1,\ 2,\ \ldots.$$

Example satisfying Wang's Lemma is Burkholder's function:

$$U_p(x,y) = p\left(1-\frac{1}{p}\right)^{p-1}((p-1)|y|-|x|)(|x|+|y|)^{p-1}, \qquad 1<p<2$$

Lemma

The function also has ($1 < p < 2$)

$$U_p(x,y) \geqslant (p-1)^p|y|^p - |x|^p + \left(1 - p\left(1-\frac{1}{p}\right)^{p-1}\right)\frac{((p-1)|y|-|x|)^2}{(|x|+|y|)^{2-p}}$$

$$\left(1-p\left(1-\frac{1}{p}\right)^{p-1}\right)\mathbb{E}\frac{((p-1)|Y_\infty|-|X_\infty|)^2}{(|X_\infty|+|Y_\infty|)^{2-p}} \leqslant ||X||_p^p-(p-1)^p||Y||_p^p$$
$$\leqslant \left(1-(1-(p-1)\varepsilon)^p\right)||X||_p^p$$
$$\leqslant p(p-1)\varepsilon||X||_p^p.$$

This, Hölder inequality, and Burkholder's estimate yield

$$||(p-1)|Y_\infty|-|X_\infty|||_p \leqslant \left(\mathbb{E}\left\{\frac{((p-1)|Y_\infty|-|X_\infty|)^2}{(|X_\infty|+|Y_\infty|)^{2-p}}\right\}\right)^{1/2}|||X_\infty|+|Y_\infty|||_p^{\frac{(2-p)}{2}}$$
$$\leqslant \left(\frac{p(p-1)\varepsilon}{1-p\left(1-\frac{1}{p}\right)^{p-1}}\right)^{1/2}||X||_p^{p/2}\cdot\left(\frac{p}{p-1}||X||_p\right)^{\frac{(2-p)}{2}}.$$

$$U_p(x,y) = \begin{cases} p\left(1-\frac{1}{p}\right)^{p-1}(|y|-(p-1)|x|)(|x|+|y|)^{p-1} & \text{if } |y| \geqslant (p-2)|x|, \\ -\frac{(p-1)^{2p-2}}{p^{p-2}}|x|^p & \text{if } |y| < (p-2)|x|. \end{cases}$$

Lemma (U_p satisfies Wang's Lemma and)

$$U_p(x,y) \geqslant |y|^p - (p-1)^p|x|^p + \alpha_p \big||y|-(p-1)|x|\big|^p,$$

$$\alpha_p = \frac{p-2}{p-1}\left(\frac{1}{2}-\frac{1}{e}\right).$$

$$\begin{aligned} \alpha_p \big\|\,|Y_\infty| - (p-1)|X_\infty|\,\big\|_p^p &\leqslant (p-1)^p\|X\|_p^p - \|Y\|_p^p \\ &\leqslant \left[(p-1)^p - (p-1-\varepsilon)^p\right]\|X\|_p^p \\ &\leqslant p(p-1)^{p-1}\varepsilon\|X\|_p^p. \end{aligned}$$

Thank You!