NEW RESULTS IN HARMONIC ANALYSIS and STOCHASTIC ANALYSIS ON MATRICES Applications of SQUARED BESSEL PARTICLE SYSTEMS

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May 19, 2017

Będlewo 3rd Conference Probability and Analysis, 2017

Articles

P. Graczyk, J. Małecki, E. Mayerhofer
A Characterization of Wishart Processes and Wishart Distributions
to appear in Stoch.Proc. Appl., 2017

P. Graczyk, J. Małecki
On Squared Bessel particle systems and matrix Yamada-Watanaba theorem,
in preparation

Mayerhofer's Conjecture on the non-central Gindikin set

Harmonic analysis and statistics on symmetric matrices: Mayerhofer's Conjecture on the non-central Gindikin set

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Usual definition: W_0 is the set of admissible $\beta \in \mathbb{R}$ such that there exists a positive Riesz measure ν, i.e. $\mathcal{L}(\nu)(u) = (\det u)^{-\beta}$

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Let $\xi_i \sim N(0, \Sigma/2)$ be independent normal vectors in $\mathbb{R}^p$ and

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Proof:
Book: Faraut-Koranyi *Analysis on Symmetric Cones*, 1994

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Then X has a probability distribution called the non-central Wishart distribution $\Gamma_p(\beta, \omega; \Sigma)$ on $\bar{\mathcal{S}}_p^+$.

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First complete proof of the Mayerhofer Conjecture :
P. Graczyk, J. Małecki, E. Mayerhofer, SPA 2017, based on Itô stochastic calculus and theory of affine processes.

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- eigenvalue processes λ_i of X are called Squared Bessel Particle Systems. If X_0 has no multiple eigenvalues,

$$d\lambda_i = 2\sqrt{\lambda_i} dB_i + \left(\alpha + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt$$

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Laplace transform of $X_t \Rightarrow$
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Proof. Let a random variable X on $\bar{\mathcal{S}}_p^+$ exist with law $\Gamma_p(\beta, \omega; \Sigma)$,
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(Use recent theory of affine processes on matrices of any rank, Cuchiero, Teichmann 2013)

Description of the stochastic Gindikin set

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Corollary. Mayerhofer's Conjecture is true.

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$$dX_t = \sqrt{X_t}dW_t + dW_t^T\sqrt{X_t} + \alpha I dt,\ X_t \in \bar{\mathcal{S}}_p^+,\ t \geq 0.$$

has a global weak solution with $X_0 \in x_0$ *if and only if*

$$(\alpha \in [p-1,\infty), x_0 \in \bar{\mathcal{S}}_p^+) \text{ or } (\alpha \in B = \{0,1,\cdots,p-2\}, rk(x_0) \leq \alpha).$$

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and follows from the quadratic construction $X_t = q(N_t)$ in the case $\alpha \in B$.

Proof of necessity in Stochastic Gindikin set : use of symmetric polynomials on BESQ particles

If X is a symmetric $p \times p$ matrix, we define the polynomials $e_n(X)$ as elementary symmetric polynomials

$$e_n(X) = \sum_{i_1 < \ldots < i_n} \lambda_{i_1}(X)\lambda_{i_2}(X)\ldots\lambda_{i_n}(X), \qquad n = 1, \ldots, p;$$

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In particular, $e_1(X) = \mathbf{Tr}(X)$ and $e_p(X) = \det X$.

SDEs for symmetric polynomials: obtained by Itô calculus on BESQ particles

Proposition. The symmetric polynomials $e_n = e_n(X)$, $n = 1, \ldots, p$ are semimartingales satisfying the following system of SDEs

$$\begin{aligned} de_1 &= 2\sqrt{e_1}dV_1 + p\alpha dt, \\ de_n &= M_n(e_1, \ldots, e_p)dV_n + (p-n+1)(\alpha - n + 1)e_{n-1}dt, \quad , \\ de_p &= 2\sqrt{e_{p-1}e_p}dV_p + (\alpha - p + 1)e_{p-1}dt, \end{aligned}$$

where V_n, $n = 1, \ldots, p$ are one-dimensional Brownian motions and the functions M_n are continuous on $\mathbb{R}^p$. The processes $\mathcal{M}_n(t) := \int_0^t M_n dV_n$ are martingales.

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$$\mathbb{E}e_1(t) = e_1(0) + p\alpha \int_0^t ds = e_1(0) + p\alpha t.$$

$$\begin{aligned}\mathbb{E}e_2(t) &= e_2(0) + (p-1)(\alpha-1)\int_0^t \mathbb{E}e_1(s)ds \\ &= e_2(0) + (p-1)(\alpha-1)e_1(0)t + p(p-1)\alpha(\alpha-1)\frac{t^2}{2},\end{aligned}$$

and so on. Consequently, the coefficient of t^n in $\mathbb{E}e_n(t)$ is

$$\frac{p(p-1)\cdot\ldots\cdot(p-n+1)\cdot\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{n!}.$$

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If $\alpha \notin \mathbb{N}$ and n is the first integer greater than or equal to $\alpha+1$, then $\mathbb{E}e_n(t)$ is a polynomial with the leading coefficient negative $\Rightarrow$ $\mathbb{E}e_n(t)$ cannot stay positive for every $t > 0$, contradiction!

Proof of necessity in Stochastic Gindikin set

If $\alpha = m \in B$, consider $\mathbb{E}e_n(t)$ where $n = m + 1$. Then

$$\mathbb{E}e_n(t) = e_n(0) + (p - n + 1)(\alpha - n + 1) \int_0^t \mathbb{E}e_{n-1}(s)ds = e_n(0).$$

If $e_n(0) > 0$, then

$$\mathbb{E}e_{n+1}(t) = e_{n+1}(0) + (p - n)(\alpha - n)e_n(0)t,$$

i.e. the leading term is negative and thus $\mathbb{E}e_{n+1}(t) < 0$ for large t. It implies $e_n(0) = 0$, i.e. $rank(x_0) \leq n - 1 = m = \alpha$.

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Yamada-Watanabe Theorem on $\mathbb{R}$. *Let $B(t)$ be a Brownian motion on $\mathbb{R}$. Consider the SDE with "1/2-Hölder" martingale function σ and Lipschitz drift function b*

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt$$

Then the **pathwise uniqueness of solutions holds***.*
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The exact condition on σ:
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Example. BESQ(α) SDE: $dX(t) = 2\sqrt{X(t)}dB(t) + \alpha dt$.

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Working on BESQ Particle Systems, we observed that this is not true.

Yamada-Watanabe Theorem is false in many dimensions !

Theorem. *Consider the BESQ(0) generalized matrix SDE on* $\mathcal{S}_2$ *(the vector space of symmetric* 2×2 *matrices, isomorphic to* $\mathbb{R}^3$*.)*

$$d\mathbb{Y}_t = \sqrt{|\mathbb{Y}_t|}\,d\mathbb{W}_t + d\mathbb{W}_t^T\sqrt{|\mathbb{Y}_t|}, \quad \mathbb{Y}_0 = 0, \;\; \mathbb{Y}_t \in \mathcal{S}_2. \tag{1}$$

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The SDE (1) *has* **two strong solutions.**

Proof. First strong solution is 0.

Second strong solution is constructed from eigenvalues and eigenvectors $\mathbb{Y} = \mathbb{H}\boldsymbol{\Lambda}\mathbb{H}^T$,

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \mathbb{H} = \begin{bmatrix} h_{11} & -h_{12} \\ h_{12} & h_{11} \end{bmatrix}$$

SDEs for eigenvalues and eigenvectors

Let B_1, B_2, β be 3 independent one-dimensional Brownian motions
The eigenvalues are two independent BESQ particles described by:

$$\begin{aligned} d\lambda_1 &= 2\sqrt{|\lambda_1|}dB_1 - dt, \quad \lambda_1(0) = 0, \\ d\lambda_2 &= 2\sqrt{|\lambda_2|}dB_2 + dt, \quad \lambda_2(0) = 0. \end{aligned}$$

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λ_1, λ_2 are BESQ processes of dimension -1 and $+1$ respectively

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dh_{11} &= \frac{-h_{12}}{\sqrt{|\lambda_1| + |\lambda_2|}} d\beta - \frac{1}{2}\frac{h_{11}dt}{|\lambda_1| + |\lambda_2|}, \quad h_{11}(0) = 1, \\
dh_{12} &= \frac{h_{11}}{\sqrt{|\lambda_1| + |\lambda_2|}} d\beta - \frac{1}{2}\frac{h_{12}dt}{|\lambda_1| + |\lambda_2|}, \quad h_{12}(0) = 0.
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$\mathbb{Y} = \mathbb{H}\mathbf{\Lambda}\mathbb{H}^T$ is a solution of

$$d\mathbb{Y}_t = \sqrt{|\mathbb{Y}_t|}d\mathbb{W}_t + d\mathbb{W}_t^T\sqrt{|\mathbb{Y}_t|}, \quad \mathbb{Y}_0 = 0$$

Main difficulty: Define 2×2 matrix of independent Brownian motions, i.e.

$$\mathbb{W}_t = \begin{bmatrix} W_1(t) & W_2(t) \\ W_3(t) & W_4(t) \end{bmatrix},$$

where W_i are 4 independent one-dimensional Brownian motions.
Step 1. Add a 4-th Brownian motion γ to B_1, B_2, β
Step 2. Equal $d(\mathbb{H}\mathbf{\Lambda}\mathbb{H}^T) = \sqrt{|\mathbb{Y}_t|}d\mathbb{W}_t + d\mathbb{W}_t^T\sqrt{|\mathbb{Y}_t|}, \quad \mathbb{Y}_0 = 0$
Step 3. Solve the obtained system of 4 linear equations in $dW_i, i = 1, 2, 3, 4$. Define W_i by the obtained formulas.

APPENDIX

Proposition

Let X_t be a stochastic matrix process on Sym_p and Λ_t its ordered eigenvalues, $\lambda_1(t) \leq \ldots \leq \lambda_p(t)$.
Suppose that X_t satisfies the SDE

$$dX_t = h(X_t)dW_t g(X_t) + g(X_t)dW_t^T h(X_t) + b(X_t)dt$$

where the functions $g, h, b : \mathbb{R} \to \mathbb{R}$ act spectrally on Sym_p.
If $\lambda_1(0) \leq \ldots \leq \lambda_p(0)$, then the process Λ_t is a semimartingale, satisfying for $t < T$=first collision time the SDEs system:

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)dB_i + \left(b(\lambda_i) + \sum_{j \neq i} \frac{G(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt,$$

where $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$.