Convergence of complex Biggins martingale on the phase boundary

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Branching random walk

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We define an associated branching random walk:

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For a particle v, we write $|v|$ the generation in which v is born, $X(v)$ for its displacement from its parent, and for $S(v)$ for its position. Thus we have

$$S(v) = \sum_{u \leq v} X(u).$$

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Observe that

$$\begin{aligned}\mathbb{E}[Z_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\sum_{|v|=n+1} e^{-\lambda S(v)}|\mathcal{F}_n] \\ &= \mathbb{E}[\sum_{|v|=n} \sum_{v<u,|u|=n+1} e^{-\lambda(S(v)+X(u))}|\mathcal{F}_n] = Z_n\mathbb{E}[Z_1].\end{aligned}$$

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In particular, for $m(\lambda) = \mathbb{E}\sum_i e^{-\lambda X_i}$, the process

$$W_n(\lambda) := \frac{Z_n(\lambda)}{\mathbb{E}Z_n(\lambda)} = m^{-n}(\lambda)\mathbb{E}[Z_n(\lambda)]$$

is a martingale, called Biggins' martingale.

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- What is the rate of convergence?

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- $W_n(\lambda)$ is a "toy model" for the Gaussian multiplicative chaos i.e. measures of the form

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- applications in physics

Real case

Let assume that $d = 1$ and $\theta \in \mathbb{R}$. By martingale convergence theorem $W_n(\theta) \to W(\theta)$ for some $W(\theta) \geq 0$.

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Theorem (Biggins 1977)

The following are equivalent

- $\mathbb{P}[W(\theta) = 0] = \mathbb{P}[$ *the branching process gets extinct* $]$
- $\mathbb{P}[W(\theta) = 0] < 1$
- $\mathbb{E}W(\theta) = 1$
- $\mathbb{E}[Z_1(\theta) \log^+ Z_1(\theta)] < \infty$ *and* $\theta m'(\theta)/m(\theta) < \log m(\theta)$

Complex case

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We define the moment generating function φ by

$$\varphi(t) = \mathbb{E}\sum_{|v|=1} |L(v)|^t$$

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Theorem (Biggins 1992)

Suppose that $\varphi(p) < 1$ *for some* $p \in (1, 2]$, *then* W_n *converges almost surely and in* L^p *to some limit* W.

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Theorem (K, Meiners)

Suppose that $\varphi(\alpha) = 1$ *and* $\varphi'(\alpha) = 0$ *for some* $\alpha \in (1, 2)$ *then* W_n *converges almost surely and in* L^p *for any* $p < \alpha$*.*
If $\varphi(2) = 1$ *then* W_n *does not converges in probability.*

Gaussian splitting

Let $\mathcal{L} = \{X_1, X_2\}$ where X_i are independent standard normal variables. In this case $m(\lambda) = 2e^{\lambda^2/2}$ and $\phi(t) = 2^{1-t}e^{\frac{t}{2}((t+1)\theta^2-\eta^2)}$

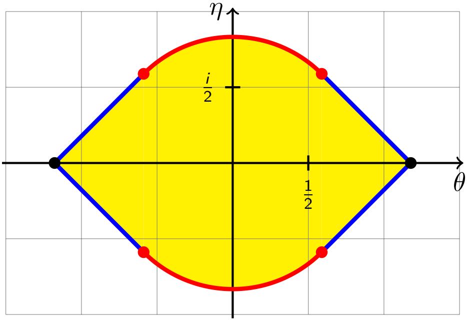

Figure: In the above example the yellow part corresponds to those $\lambda = \theta + i\eta$ for which $\phi(p) < 1$ for some $p \in (1, 2]$. The blue lines corresponds to the case $\phi(\alpha) = 1$ and $\phi'(\alpha) = 0$ for some $\alpha \in (1, 2)$. The red curve corresponds to the case $\phi(2) = 1$.

Change of the measure

Suppose that

$$\mathbb{E}\sum_{|v|=1}|L(v)|^{\alpha}=1.$$

Then we can define a random variable Y by

$$\mathbb{E}[f(Y)]=\mathbb{E}\sum_{|v|=1}f(\log|L(v)|)|L(v)|^{\alpha},$$

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Many to one lemma

For any function $F : \mathbb{R}^n \mapsto \mathbb{R}$ we have

$$\mathbb{E}\left[\sum_{|v|=n} F(S(v_0), S(v_1), \ldots, S(v_n)) |L(v)|^{\alpha}\right] = \mathbb{E} F(S_0, S_1, \ldots, S_n),$$

where v_i is an ancestor of v from the i–th generation.

Idea of the proof: Naive approach

First we can try to estimate the second moment of the martingale:

$$\begin{aligned}
\mathbb{E}|W_n - 1|^2 &= \sum_{i \le n-1} \mathbb{E}|W_{i+1} - W_i|^2 \\
&= \sum_{i \le n-1} \mathbb{E} \sum_{|v|=i} |L(v)|^2 \left| \sum_{u:u'=v} L(v)^{-1} L(u) - 1 \right|^2 \\
&= \mathbb{E} \left| \sum_{|u|=1} L(u) - 1 \right|^2 \mathbb{E} \sum_{|v| \le n-1} |L(v)|^2 \\
&= C(1 + \phi(2) + \cdots + \phi^{n-1}(2)),
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Since $\phi(2) > 1$ the above sum diverges.
Reason: $\max_v |L(v)|$ has too heavy tail.

Idea of the proof: estimation of the maximum

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We have

$$\mathbb{P}[\max_v L(v) > t] \leq \mathbb{E}[\sharp\{v : L(v) > t,\ L(u) \leq t \text{ for all } u \leq v\}]$$
$$= \sum_v \mathbb{E}1_{[L(v)>t,\ L(u)\leq t \text{ for all } u\leq v]}$$
$$= \sum_n \mathbb{E}e^{-\alpha S_n} 1_{[S_n > \log t,\ S_k \leq t \text{ for all } k \leq n]} \leq t^{-\alpha}$$

Idea of the proof: truncation argument

For given t, we define a new martingale $\{W_n^t\}$ by $W_0^t = 1$ and

$$W_{i+1}^t - W_i^t = \sum_{|v|=i} L(v) 1_{[|L(w)|\leq t \text{ for all } w\leq v]} \left(\sum_{u:u'=v} L(v)^{-1} L(u) - 1 \right).$$

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We can compute the second moment:

$$\begin{aligned} \mathbb{E}|W_n^t - 1|^2 &= \sum_{i \le n-1} \mathbb{E}|W_{i+1}^t - W_i^t|^2 \\ &= C \sum_{i \le n-1} \mathbb{E}|L(v)|^2 1_{[|L(w)| \le t \text{ for all } w \le v]} \\ &= C \sum_{i \le n-1} \mathbb{E} e^{(2-\alpha) S_i} 1_{[S_j \le \log t \text{ for all } j \le i]} \lesssim t^{2-\alpha} \end{aligned}$$

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In particular, W_n is L^p bounded martingale for any $p < \alpha$. □

Rate of convergence (work in progress)

We have the following tail behavior

Theorem (Iksanov, K, Meiners)

For the limit W of the martingale W_n we have

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As a consequence we obtain

Theorem (Ikasnov, K, Meiners)

$$n^{\frac{1}{2\alpha}}(W - W_n) \xrightarrow{d} \sqrt[\alpha]{D_\infty} S_\alpha,$$

where D_∞ is the limit of derivative martingale and S_α is independent α stable distrbution.