A Noncommutative Catenoid

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References

Noncommutative Minimal Surfaces. Lett. Math. Phys, 2016.
J. A., J. Hoppe and J. Choe.

A Noncommutative Catenoid. `arXiv:1706.10168`
J. A. and C. Holm.

Why noncommutative minimal surfaces?

The classical theory of minimal surfaces is an old and rich subject, and still quite active.

From a mathematical point of view, it is interesting to investigate if one can develop a parallel theory for noncommutative geometry.

There are many explicit examples of minimal surfaces that can be turned into noncommutative ones. In this way, one can provide a multitude of examples of noncommutative surfaces.

Analogues of minimal surface equations appear as equations of motion in physical models; e.g. in Membrane and String theory one finds that the (operators corresponding to the) embedding coordinates have to be harmonic. For physicists, the following equation might look familiar:

$$\sum_{k=1}^{n} [[X^i, X^k], X^k] = 0.$$

Minimal surfaces in Euclidean space

Let $\Omega \subseteq \mathbb{R}^2$ such that $\vec{x} : \Omega \to \mathbb{R}^n$ describes a surface Σ in $\mathbb{R}^n$. Classically, $\vec{x} : \Omega \to \mathbb{R}^n$ is called a *minimal surface* if it is a stationary point of the area integral:

$$A[\vec{x}] = \int \sqrt{g}\, du dv$$

where g denotes the induced metric on Σ.This is equivalent to demanding that the embedding coordinates x^i are harmonic; i.e.

$$\Delta_\Sigma(x^i) = 0 \qquad \text{for } i = 1, 2, \ldots, n,$$

where Δ_Σ denotes the Laplace-Beltrami operator on Σ. (There are of course other characterizations.)

Poisson algebraic formulation of geometry

Assume that Σ is a 2-dimensional manifold, with local coordinates $u = u^1, v = u^2$, embedded in $\mathbb{R}^n$ via the embedding coordinates $x^1(u,v), x^2(u,v), \ldots, x^n(u,v)$, inducing on Σ the metric

$$g_{ab} = \partial_a \vec{x} \cdot \partial_b \vec{x} \equiv \sum_{i=1}^{n} \left(\partial_a x^i\right)\left(\partial_b x^i\right)$$

where $\partial_a = \frac{\partial}{\partial u^a}$. We adopt the convention that indices a, b, p, q take values in $\{1,2\}$, and i, j, k, l run from 1 to n.
For an arbitrary density ρ, one may introduce a Poisson bracket on $C^\infty(\Sigma)$ via

$$\{f, h\} = \frac{1}{\rho}\varepsilon^{ab}\left(\partial_a f\right)\left(\partial_b h\right),$$

and we define the function $\gamma = \sqrt{g}/\rho$, where g denotes the determinant of the metric g_{ab}.

Setting $\theta^{ab} = \frac{1}{\rho}\varepsilon^{ab}$ (the Poisson bivector) one notes that

$$\theta^{ap}\theta^{bq}g_{pq} = \frac{1}{\rho^2}\varepsilon^{ap}\varepsilon^{bq}g_{pq} = \frac{g}{\rho^2}g^{ab} = \gamma^2 g^{ab} \tag{1}$$

since $\varepsilon^{ap}\varepsilon^{bq}g_{pq}$ is the cofactor expansion of the inverse of the metric. The fact that the geometry of the submanifold Σ can be expressed in terms of Poisson brackets follows from the trivial, but crucial, observation that the projection operator $\mathcal{P} : T\mathbb{R}^n \to T\Sigma$ (where one regards $T\Sigma$ as a subspace of $T\mathbb{R}^n$) can be written as

$$\mathcal{P}(X)^i = \frac{1}{\gamma^2}\sum_{j,k=1}^{n}\{x^i, x^k\}\{x^j, x^k\}X^j$$

for $X \in T\mathbb{R}^n$. (It follows from relation (1).)

Multi linear formulation of differential geometry and matrix regularizations (Arnlind, Hoppe, Huisken, J. Diff. Geom, 2012)
Pseudo-Riemannian geometry in terms of multi-linear brackets (Arnlind, Huisken, Lett. Math. Phys, 2014)

In the same spirit, one can easily show that the Laplace-Beltrami operator on Σ can be written in the following two forms

$$\Delta(f) = \gamma^{-1}\sum_{i=1}^{n}\{\gamma^{-1}\{f, x^i\}, x^i\}$$

$$\Delta(f) = \gamma^{-1}\{\gamma^{-1}\{f, u^a\}g_{ab}, u^b\}.$$

On a surface, one may always find *conformal coordinates*; i.e., coordinates with respect to which the metric becomes $g_{ab} = \mathcal{E}(u, v)\delta_{ab}$ for some (strictly positive) function $\mathcal{E}$. Furthermore, if we choose $\rho = 1$ (giving $\gamma = \mathcal{E}$), the second formula above can be written as

$$\Delta(f) = \frac{1}{\mathcal{E}}\{\{f, u^a\}\delta_{ab}, u^b\} = \frac{1}{\mathcal{E}}\{\{f, u\}, u\} + \frac{1}{\mathcal{E}}\{\{f, v\}, v\}$$

Minimal surfaces can be characterized by the fact that their embedding coordinates $x^1, \ldots, x^n$ are harmonic with respect to the Laplace operator on the surface; i.e. $\Delta(x^i) = 0$ for $i = 1, \ldots, n$. In local conformal coordinates, due to the above Poisson algebraic formulas, one may formulate this as follows:

A surface $\vec{x} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ is minimal if

$$\Delta_0(x^i) = \{\{x^i, u\}, u\} + \{\{x^i, v\}, v\} = 0 \text{ for } i = 1, \ldots, n$$
$$\vec{x}_u \cdot \vec{x}_u = \vec{x}_v \cdot \vec{x}_v \text{ and } \vec{x}_u \cdot \vec{x}_v = 0$$

We also note that the above Poisson bracket satisfies $\{u, v\} = 1$. These formulas make up our starting point when generalizing to noncommutative algebras.

We were aiming to solve equations.

The Weyl algebra

In the geometrical setting, we introduced a Poisson bracket with $\{u, v\} = 1$. Therefore, we shall be interested in a (noncommutative) unital algebra containing two elements U, V satisfying

$$[U, V] = i\hbar\mathbb{1},$$

for some real number $\hbar > 0$. The associative unital algebra generated by U, V satisfying the above relation is commonly referred to as the *Weyl algebra*.

The Weyl algebra satisfies the so called *Ore condition*, which implies that it can be embedded in a field of fractions by a general procedure. By $\mathcal{A}_\hbar$ we shall denote the Weyl algebra, and by $\mathfrak{F}_\hbar$ its field of fractions.

Derivations

Let us introduce the inner derivations:

$$\partial_u(A) = \frac{1}{i\hbar}[A, V]$$
$$\partial_v(A) = -\frac{1}{i\hbar}[A, U],$$

from which it follows that $\partial_u\big(\partial_v(A)\big) = \partial_v\big(\partial_u(A)\big)$.
Compare with the geometric setting (with the choice $\rho = 1$), where it holds that

$$\frac{\partial f}{\partial u} = \{f, v\} \quad \text{and} \quad \frac{\partial f}{\partial v} = -\{f, u\}.$$

We also set $\Lambda = U + iV$ as well as

$$\partial = \frac{1}{2}(\partial_u - i\partial_v) \quad \text{and} \quad \bar{\partial} = \frac{1}{2}(\partial_u + i\partial_v).$$

We shall construct minimal surfaces by finding "embedding coordinates"

$$X = (X^1, \ldots, X^n) \in \mathfrak{F}_\hbar^n$$

that are harmonic. Then we define the noncommutative minimal surface as the algebra generated by $X^1, \ldots, X^n$ (and closed w.r.t. ∂_u, ∂_v).

In analogy with the geometric equations, we demand

$$\big[[X^i, V], V\big] + \big[[X^i, U], U\big] = 0$$

as well as the "conformal" condition

$$h(\partial_u X, \partial_u X) = h(\partial_v X, \partial_v X)$$
$$h(\partial_u X, \partial_v X) + h(\partial_v X, \partial_u X) = 0$$

where $h : \mathfrak{F}_\hbar^n \times \mathfrak{F}_\hbar^n \to \mathfrak{F}_\hbar$ is the hermitian form

$$h(U, V) = \sum_{i=1}^{n} (U^i)^* V^i.$$

All of this can be put in a much more polished form, but as our original emphasis lied on the equations, I wanted to give you a down to earth presentation.

Now, is the above definition of a noncommutative minimal surface useful? The idea of introducing minimal surfaces in this way, was to see to what extent one may generalize the classical minimal surfaces in $\mathbb{R}^3$ to noncommutative ones.

Surprisingly, one can prove an analogy of the Weierstrass representation theorem. Recall that this theorem completely characterizes all minimal surfaces in $\mathbb{R}^3$. Moreover, it gives an explicit way to construct infinitely many minimal surfaces.

The noncommutative Weierstrass representation theorem

Theorem

Let $\{X^1, X^2, X^3\}$ *be a minimal surface for which it holds that* $\partial(X^1 - iX^2) \neq 0$*. Then there exist r-holomorphic elements* $f, g \in \mathfrak{F}_\hbar$ *together with* $x^i \in \mathbb{R}$ *(for* $i = 1, 2, 3$*), such that*

$$\begin{aligned} X^1 &= x^1\mathbb{1} + \mathrm{Re}\int \frac{1}{2}f(\mathbb{1} - g^2)d\Lambda \\ X^2 &= x^2\mathbb{1} + \mathrm{Re}\int \frac{i}{2}f(\mathbb{1} + g^2)d\Lambda \\ X^3 &= x^3\mathbb{1} + \mathrm{Re}\int fgd\Lambda. \end{aligned} \tag{2}$$

Conversely, for any r-holomorphic f *and* g *such that* $f(1 - g^2)$*,* $f(1 + g^2)$ *and* fg *are integrable, the above equations define a minimal surface.*

Algebraic minimal surfaces

As an example, one may generalize the following class of algebraic minimal surfaces in $\mathbb{R}^3$ to the noncommutative setting. Namely, for each polynomial $F(\Lambda)$, the following defines a noncommutative minimal surface:

$$X^1 = \mathrm{Re}\left((\mathbb{1} - \Lambda^2)\partial^2 F(\Lambda) + 2\Lambda\partial F(\Lambda) - 2F(\Lambda)\right)$$

$$X^2 = \mathrm{Re}\left(i(\mathbb{1} + \Lambda^2)\partial^2 F(\Lambda) - 2i\Lambda\partial F(\Lambda) + 2iF(\Lambda)\right)$$

$$X^3 = \mathrm{Re}\left(2\Lambda\partial^2 F(\Lambda) - 2\partial F(\Lambda)\right)$$

For instance, choosing $F(\Lambda) = \Lambda^3$ one obtains a noncommutative version of the Enneper surface

$$X^1 = U + UV^2 - \frac{1}{3}U^3 - i\hbar V$$

$$X^2 = -V - U^2 V + \frac{1}{3}V^3 + i\hbar U$$

$$X^3 = U^2 - V^2.$$

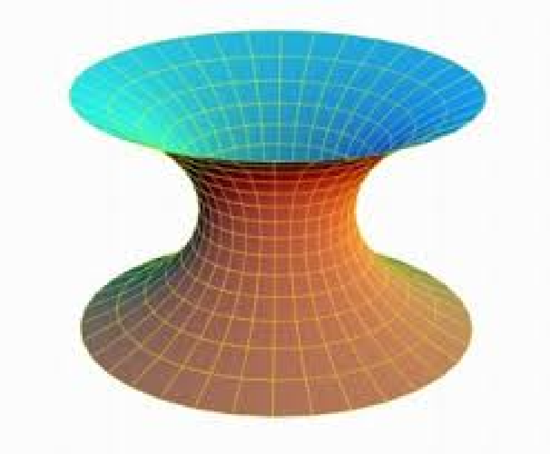
Other examples?

Thus, it turns out that one may directly generate noncommutative analogues of many minimal surfaces in $\mathbb{R}^3$. Of course, if we stay in the polynomial algebra generated by U, V, only algebraic ones may be constructed. However, if we work with representations or an extension of the algebra, then more examples may be obtained.

In the following, a noncommutative catenoid will be constructed, and its Riemannian aspects will be investigated, as well as a large class of bimodules over the algebra.

The catenoid

Catenoid – a minimal surface connecting two circles.



One radial (compact) direction, and one non-compact direction.

The catenoid

Recall the catenoid as a parametrized minimal surface in $\mathbb{R}^3$:

$$\vec{x}(u,v) = \big(\cosh(u)\cos(v), \cosh(u)\sin(v), u\big)$$

for $-\infty < u < \infty$ and $0 \leq v \leq 2\pi$, and the embedding coordinates are no longer polynomials in u and v. The polynomial algebra generated by the functions x^1, x^2, x^3 can also be generated by u, $e^{\pm u}$ and $e^{\pm iv}$.

Let us therefore try to construct an algebra “generated by”

$$U \qquad e^{\pm U} \qquad e^{\pm iV}$$

(of course, these are only formal symbols).

A Noncommutative Catenoid $\mathcal{C}_\hbar$

We let $\mathbb{C}\left\langle U, R^{\pm 1}, W^{\pm 1}\right\rangle$ be the free unital algebra generated by these symbols where we keep in mind that

$$R^{\pm 1} \sim e^{\pm U} \qquad W^{\pm 1} \sim e^{\pm iV}$$

Moreover, we introduce a $*$-algebra structure via $U^* = U$, $R^* = R$ and $W^* = W^{-1}$, as well as the relations

$$WR = e^{\hbar} RW, \quad WU = UW + \hbar W \quad \text{and} \quad [U, R] = 0.$$

(together with the corresponding $*$-versions). Formally, they can be "derived" by using $[U, V] = i\hbar\mathbb{1}$; e.g.

$$\begin{aligned} WR &= e^{iV}e^{U} = e^{iV+U+\frac{1}{2}[iV,U]} = e^{U+iV+\frac{1}{2}\hbar} = e^{\hbar}e^{U+iV-\frac{1}{2}\hbar} \\ &= e^{\hbar}e^{U+iV+\frac{1}{2}[U,iV]} = e^{\hbar}e^{U}e^{iV} = e^{\hbar}RW. \end{aligned}$$

The algebra of the noncommutative catenoid will be denoted by $\mathcal{C}_\hbar$.

A basis for $\mathcal{C}_\hbar$

By using the "Diamond lemma" one can prove that the algebra is indeed non-trivial and a basis for $\mathcal{C}_\hbar$ is given by

$$U^i R^j W^k$$

for $i \geq 0$ and $j, k \in \mathbb{Z}$.

Derivations

One can show that there exist hermitian derivations ∂_u, ∂_v such that $[\partial_u, \partial_v] = 0$ and

$$\begin{aligned} \partial_u U &= \mathbb{1} & \partial_u R &= R & \partial_u W &= 0 \\ \partial_v U &= 0 & \partial_v R &= 0 & \partial_v W &= iW \end{aligned}$$

As usual, we set

$$\begin{aligned} \partial &= \frac{1}{2}(\partial_u - i\partial_v) \\ \bar{\partial} &= \frac{1}{2}(\partial_u + i\partial_v). \end{aligned}$$

Finally, we let $\mathfrak{g}$ denote the abelian complex Lie algebra generated by ∂ and $\bar{\partial}$.

Field of fractions

Proposition

The algebra $\mathcal{C}_\hbar$ has no zero-divisors.

Proposition

$\mathcal{C}_\hbar$ satisfies the Ore condition.

These results imply that there exists a total field of fractions of $\mathcal{C}_\hbar$ and that the injection of $\mathcal{C}_\hbar$ into the fraction field is injective.

Of course, we do not want to consider the fraction field as an algebra of global functions, since there are many elements that should not be invertible.

Adding more functions

The algebra generated by $U, R^{\pm 1}, W^{\pm 1}$ contain only a few of the functions on the catenoid. Since the typical representations of the algebra are given by unbounded operators, there is no natural way to make this into a C^*-algebra. One may extend the algebra in several different ways to include more smooth functions. Let us do the following.

Let $Z_\hbar(U, R)$ denote the commutative subalgebra of $\mathcal{C}_\hbar$ generated by $\mathbb{1}$, U, R, R^{-1}. Let us define a homomorphism (of commutative algebras) $\phi : Z_\hbar(U, R) \to C^\infty(\Sigma)$ via

$$\phi(\mathbb{1}) = 1 \qquad \phi(U) = u \qquad \phi(R) = e^u \qquad \phi(R^{-1}) = e^{-u}.$$

Define the following subset of $Z_\hbar(U, R)$:

$$Z_\hbar^+(U, R) = \{p \in Z_\hbar(U, R) : |\phi(p)(u)| > 0 \text{ for all } u \in \mathbb{R}\}.$$

Adding more functions

Lemma

$Z_\hbar^+(U,R)$ *is a multiplicative set.*

Given any subset S of an algebra, one may always construct an algebra (a "localization") where elements of S are invertible. However, a priori, there is no information on the kernel of the injection map into the localization (it might be the whole algebra).

However, due to the fact that we have proven that the Ore condition holds, and that there are no zero divisors one may conclude that the injection map is injective. Hence, the localization

$$\widehat{\mathcal{C}}_\hbar = Z_\hbar^+(U,R)^{-1}\mathcal{C}_\hbar$$

is non-trivial. That is, in $\widehat{\mathcal{C}}_\hbar$ polynomials that are classically non-zero are invertible. E.g. $1+R+U^2$ is invertible in $\widehat{\mathcal{C}}_\hbar$.

Adding more functions

Lemma

For every $p \in Z_\hbar^+(U,R)$ there exists $q \in Z_\hbar^+(U,R)$ such that

$$Wp = qW$$

where $q(U,R) = p(U + \hbar\mathbb{1}, e^\hbar R)$.

Using the above result, one proves that every element of $a \in \widehat{\mathcal{C}}_\hbar$ can be written as

$$a = \sum_{k\in\mathbb{Z}} a_k W^k$$

where a_k is a quotient of two polynomials in $U, R^{\pm 1}$.

Integration

The total integral of a function on the catenoid, with respect to the induced metric can be computed in local coordinates as

$$\tau(f) = \int_{-\infty}^{\infty} \left(\int_{0}^{2\pi} f(u,v) \cosh^2(u) dv \right) du$$

whenever the integral exists.For a function, expressible as

$$f(u,v) = \sum_{k \in \mathbb{Z}} f_k(u, e^u) e^{ikv}$$

we formally get (only the $k = 0$ term survives)

$$\tau(f) = 2\pi \int_{-\infty}^{\infty} f_0(u, e^u) \cosh^2(u) du.$$

Integration

Let us introduce a noncommutative integral in analogy with the classical situation. Thus, given

$$a = \sum_{k\in\mathbb{Z}} a_k(U, R^{\pm 1})W^k$$

set

$$\tau(a) = 2\pi \int_{-\infty}^{\infty} \phi(a_0)\cosh^2(u)du$$

whenever the integral exists. (Recall: $\phi(U) = u$, $\phi(R^{\pm 1}) = e^{\pm u}$.)

τ is not a trace: (formal computation!)

$$\tau(WRW^{-1}) = e^{\hbar}\tau(R) \neq \tau(R)$$

(One can take a real example where the integrals converge.)

Curvature

For the classical catenoid, parametrized by

$$\vec{x}(u,v) = \big(\cosh(u)\cos(v), \cosh(u)\sin(v), u\big)$$

the module of (complex) vector fields can be spanned by ϕ and $\bar{\phi}$, where

$$\phi = (\sinh(z), -i\cosh(z), 1) \qquad (z = u + iv).$$

Let $\{e_1, e_2, e_3\}$ denote the canonical basis of the free (right) module $(\widehat{\mathcal{C}}_\hbar)^3$, and let $h : (\widehat{\mathcal{C}}_\hbar)^3 \times (\widehat{\mathcal{C}}_\hbar)^3 \to \widehat{\mathcal{C}}_\hbar$ denote the bilinear hermitian form defined by

$$h(X,Y) = \sum_{i=1}^{3} (X^i)^* Y^i$$

for $X = e_i X^i$ and $Y = e_i Y^i$.

Module of vector fields

For the noncommutative catenoid $\widehat{\mathcal{C}}_\hbar$ we introduce

$$\Phi^1 = \frac{1}{2}e^{\frac{1}{2}\hbar}(RW - R^{-1}W^{-1}) \sim \frac{1}{2}(e^u e^{iv} - e^{-u}e^{-iv}) = \sinh(z)$$
$$\Phi^2 = -\frac{i}{2}e^{\frac{1}{2}\hbar}(RW + R^{-1}W^{-1})$$
$$\Phi^3 = \mathbb{1},$$

and set

$$\Phi = e_1\Phi^1 + e_2\Phi^2 + e_3\Phi^3$$
$$\bar{\Phi} = e_1(\Phi^1)^* + e_2(\Phi^2)^* + e_3(\Phi^3)^*,$$

and let $\mathcal{X}(\widehat{\mathcal{C}}_\hbar)$ denote the (right) $\widehat{\mathcal{C}}_\hbar$-module generated by Φ and $\bar{\Phi}$.

Proposition

$\mathcal{X}(\widehat{\mathcal{C}}_\hbar)$ is a free (right) $\widehat{\mathcal{C}}_\hbar$-module of rank 2.

Connections

A connection on $\mathcal{X}(\widehat{\mathcal{C}}_\hbar)$ is called *hermitian* if

$$dh(X, Y) = h(\nabla_{d^*}X, Y) + h(X, \nabla_d Y),$$

for $d \in \mathfrak{g}$ and $X, Y \in \mathcal{X}(\widehat{\mathcal{C}}_\hbar)$. Moreover, we say that ∇ is *torsion-free* if

$$\nabla_\partial \bar{\Phi} = \nabla_{\bar{\partial}} \Phi.$$

Let us introduce an almost complex structure $J : \mathcal{X}(\widehat{\mathcal{C}}_\hbar) \to \mathcal{X}(\widehat{\mathcal{C}}_\hbar)$

$$J\Phi = i\Phi$$
$$J\bar{\Phi} = -i\bar{\Phi}$$

and extending J to $\mathcal{X}(\widehat{\mathcal{C}}_\hbar)$ as a (right) $\widehat{\mathcal{C}}_\hbar$-module homomorphism. A connection is called *almost complex* if

$$(\nabla_d J)(X) \equiv \nabla_d J(X) - J\nabla_d X = 0$$

for all $d \in \mathfrak{g}$ and $X \in \mathcal{X}(\widehat{\mathcal{C}}_\hbar)$.

Levi-Civita connection

Introduce a metric on $\mathcal{X}(\widehat{\mathcal{C}}_\hbar)$ via

$$S = h(\Phi, \Phi), \quad T = h(\bar{\Phi}, \bar{\Phi}) \quad \text{and} \quad h(\Phi, \bar{\Phi}) = 0.$$

Theorem

There exists a unique hermitian torsion-free almost complex connection ∇ on $\mathcal{X}(\widehat{\mathcal{C}}_\hbar)$, given by

$$\nabla_\partial \Phi = \Phi S^{-1} \partial S$$
$$\nabla_{\bar\partial} \bar{\Phi} = \bar{\Phi} T^{-1} \bar{\partial} T$$
$$\nabla_{\bar\partial} \Phi = \nabla_\partial \bar{\Phi} = 0.$$

Curvature

The curvature $R(\partial_a, \partial_b) = \nabla_a\nabla_b - \nabla_b\nabla_a$ is easily computed to be

$$R(\partial, \bar\partial)\Phi = -\Phi\bar\partial\big(S^{-1}\partial S\big)$$
$$R(\partial, \bar\partial)\bar\Phi = \bar\Phi\partial\big(T^{-1}\bar\partial T\big)$$

and since $\mathcal{X}(\widehat{\mathcal{C}}_\hbar)$ is a free module, one has uniquely defined curvature components $R(\partial_a, \partial_b)\Phi_c = \Phi_p R^p{}_{cab}$ given by

$$R^1{}_{112} = -\bar\partial\big(S^{-1}\partial S\big) \qquad R^2{}_{212} = \partial\big(T^{-1}\bar\partial T\big)$$
$$R^1{}_{212} = R^2{}_{112} = 0.$$

One may also proceed to define $R_{abpq} = h(\bar\Phi_a, R(\partial_p, \partial_q)\Phi_b)$, where $\bar\Phi_1 = \bar\Phi$ and $\bar\Phi_2 = \Phi$, giving

$$R_{1212} = T\partial\big(T^{-1}\bar\partial T\big) \qquad R_{2112} = -S\bar\partial\big(S^{-1}\partial S\big)$$
$$R_{1112} = R_{2212} = 0$$

Pseudo-Riemannian calculi

The procedure of constructing a calculus over the catenoid by choosing a "module of vector fields" and associate derivations

$$\partial \leftrightarrow \Phi \qquad \bar{\partial} \leftrightarrow \bar{\Phi}$$

can be put in a more general context.

Riemannian curvature of the noncommutative 3-sphere
J. A. and M. Wilson. J. Noncommut. Geo. 2017

On the Chern-Gauss-Bonnet theorem for the noncommutative 4-sphere
J. A. and M. Wilson. J. Geom. Phys. 2016.

In the setting of "Pseudo-Riemannian calculi" one may discuss Levi-Civita connections and their corresponding curvature.

Computing the total curvature

Let us check a Gauss-Bonnet-type theorem for the catenoid. Assume that we have a conformal metric of the type $H = h(\Phi, \Phi) = h(\bar{\Phi}, \bar{\Phi})$ and $h(\bar{\Phi}, \Phi) = 0$ with H an invertible rational function in U and R.

The determinant of the metric corresponds to H again, and we may introduce

$$\tau_H(a) = 2\pi \int_{-\infty}^{\infty} \phi(a_0) h \, du$$

where $h = \phi(H)$. The Gaussian curvature is

$$K = \frac{1}{2} h^{ab} R_{apbq} h^{ab} = \hat{\partial}_u \big(H^{-1} \hat{\partial}_u H \big) H^{-1}$$

giving

$$\tau_H(K) = 2\pi \int_{-\infty}^{\infty} \partial_u (h^{-1} \partial_u h) du$$

Computing the total curvature

Is the integral independent of h?

$$\tau_H(K) = 2\pi \int_{-\infty}^{\infty} \partial_u(h^{-1}\partial_u h)du$$

Let $h = e^{f(u)} h_0(u) = e^{f(u)} \cosh^2(u)$:

$$\begin{aligned}\tau_H(K) &= 2\pi \Big[\partial_u \ln\left(e^f h_0\right)\Big]_{-\infty}^{\infty} = 2\pi \Big[\partial_u \ln h_0\Big]_{-\infty}^{\infty} + 2\pi \Big[\partial_u f\Big]_{-\infty}^{\infty} \\ &= 2\pi \Big[\partial_u \ln h_0\Big]_{-\infty}^{\infty} = -4\pi\end{aligned}$$

assuming that f behaves appropriately at infinity.

Bimodules

In many ways, the algebra of the noncommutative catenoid behaves like a "noncompact noncommutative torus".

For the noncommutative torus, there is a class of explicit bimodules appearing in the classification of projective modules over the NC torus.

It may not be surprising that one can construct similar bimodules for the noncommutative catenoid.

Bimodules

We shall represent the algebra $\mathcal{C}_\hbar$ on the following space:

Let $C_0^\infty(\mathbb{R}\times\mathbb{Z})$ denote the space of complex valued smooth functions on $\mathbb{R}\times\mathbb{Z}$ with compact support and the inner product

$$\langle \xi,\eta\rangle = \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\xi(x,k)\bar{\eta}(x,k)dx.$$

Left $\mathcal{C}_\hbar$-modules

Let $\lambda_0, \lambda_1, \varepsilon \in \mathbb{R}$ and $r \in \mathbb{Z}$. If $\lambda_0\varepsilon + \lambda_1 r = -\hbar$ then

$$
\begin{aligned}
(W\xi)(x,k) &= \xi(x-\varepsilon, k-r) \\
(W^{-1}\xi)(x,k) &= \xi(x+\varepsilon, k+r) \\
(R\xi)(x,k) &= e^{\lambda_0 x + \lambda_1 k}\xi(x,k) \\
(R^{-1}\xi)(x,k) &= e^{-\lambda_0 x - \lambda_1 k}\xi(x,k) \\
(U\xi)(x,k) &= (\lambda_0 x + \lambda_1 k)\xi(x,k)
\end{aligned}
$$

for $\xi \in C_0^\infty(\mathbb{R}\times\mathbb{Z})$, defines a left $\mathcal{C}_\hbar$-module structure on $C_0^\infty(\mathbb{R}\times\mathbb{Z})$ (compatible with the $*$-structure).

Right $\mathcal{C}_\hbar$-modules

Correspondingly,

$$
\begin{aligned}
(\xi W)(x,k) &= \xi(x-\varepsilon',k-r') \\
(\xi W^{-1})(x,k) &= \xi(x+\varepsilon',k+r') \\
(\xi R)(x,k) &= e^{\mu_0 x+\mu_1 k}\xi(x,k) \\
(\xi R^{-1})(x,k) &= e^{-\mu_0 x-\mu_1 k}\xi(x,k) \\
(\xi U)(x,k) &= (\mu_0 x+\mu_1 k)\xi(x,k)
\end{aligned}
$$

defines a right $\mathcal{C}_\hbar$-module structure on $C_0^\infty(\mathbb{R}\times\mathbb{Z})$ (compatible with the $*$-structure) if $\mu_0,\mu_1,\varepsilon'\in\mathbb{R}$ and $r'\in\mathbb{Z}$ such that $\mu_0\varepsilon'+\mu_1 r'=\hbar$.

Bimodules

Bimodule conditions for a $\mathcal{C}_\hbar - \mathcal{C}_{\hbar'}$-bimodule:

$$
\begin{aligned}
&\lambda_0\varepsilon + \lambda_1 r = -\hbar && \text{(Left module)}\\
&\mu_0\varepsilon' + \mu_1 r' = \hbar' && \text{(Right module)}\\
&\lambda_0\varepsilon' + \lambda_1 r' = 0 && \text{(Bimodule)}\\
&\mu_0\varepsilon + \mu_1 r = 0 && \text{(Bimodule)}
\end{aligned}
$$

with

$$
\begin{aligned}
&\lambda_0, \lambda_1, \varepsilon, \varepsilon' \in \mathbb{R}\\
&r, r' \in \mathbb{Z}.
\end{aligned}
$$

Note that the above equations have solutions for arbitrary $\hbar, \hbar' \in \mathbb{R}$.

Connections of constant curvature

Define linear maps $\nabla_u, \nabla_v : C_0^\infty(\mathbb{R} \times \mathbb{Z}) \to C_0^\infty(\mathbb{R} \times \mathbb{Z})$ via

$$(\nabla_u \xi)(x,k) = \alpha \frac{d\xi}{dx}(x,k) \qquad \text{and} \qquad (\nabla_v \xi)(x,k) = \beta x \xi(x,k) \tag{3}$$

for $\alpha, \beta \in \mathbb{C}$. It is straightforward to check that

$$\nabla_u(a\xi) = a\nabla_u\xi + (\partial_u a)\xi$$
$$\nabla_v(a\xi) = a\nabla_v\xi + (\partial_v a)\xi$$

for all $a \in \mathcal{C}_\hbar$ if and only if $\alpha = 1/\lambda_0$ and $\beta = i/\varepsilon$. Similarly, it holds that

$$\nabla_u(\xi a) = (\nabla_u\xi)a + \xi(\partial_u a)$$
$$\nabla_v(\xi a) = (\nabla_v\xi)a + \xi(\partial_v a)$$

for all $a \in \mathcal{C}_{\hbar'}$ if and only if $\alpha = 1/\mu_0$ and $\beta = i/\varepsilon'$.

Assume that $C_0^\infty(\mathbb{R}\times\mathbb{Z})$ is a $\mathcal{C}_\hbar - \mathcal{C}_{\hbar'}$-bimodule and that

$$(\nabla_u\xi)(x,k) = \frac{1}{\lambda_0}\frac{d\xi}{dx}(x,k)$$
$$(\nabla_v\xi(x,k)) = \frac{i}{\varepsilon}x\xi(x,k)$$

is a bimodule connection on $C_0^\infty(\mathbb{R}\times\mathbb{Z})$.

1. If $\hbar=\hbar'$ then $\hbar=\hbar'=0$,
2. if $\hbar\neq\hbar'$ then $\hbar/\hbar'\in\mathbb{Q}$ and

$$\lambda_0=\mu_0=\frac{\hbar r'}{\varepsilon(r-r')} \qquad \lambda_1=-\frac{\hbar}{r-r'} \qquad \mu_1=-\frac{\hbar'}{r-r'}$$

for arbitrary $\varepsilon=\varepsilon'\in\mathbb{R}$ and $r,r'\in\mathbb{Z}$ such that $r/r'=\hbar/\hbar'$. Moreover,

$$\nabla_u\nabla_v\xi(x,k)-\nabla_v\nabla_u\xi(x,k)=i\frac{\hbar-\hbar'}{\hbar\hbar'}\xi(x,k).$$

Projective modules?

Are these modules projective? As far as I know, in order to prove that the corresponding modules for the NC torus are projective, one constructs Morita equivalence bimodules, and the projectivity is automatic once such a structure has been set up.

Unfortunately, the same type of construction doesn't immediately work because of the non-periodicity of one direction of the catenoid. However, this question together with the classification of projective modules over the noncommutative catenoid is a very interesting question.

Summary

- We have introduced noncommutative minimal surfaces from an equational point of view, where we demand the embedding coordinates to be harmonic.
- This was applied to the Weyl algebra, where one may construct infinite classes of subalgebras of the Weyl algebra by generalizing the classical Weierstrass representation theorem.
- Our hope is that these algebras enjoy properties that allows for the construction of "nice and interesting" noncommutative geometries.
- A noncommutative catenoid was constructed, and we show that the differential calculus is very similar to the NC torus, and a theory of Riemannian curvature can be introduced.
- There is a unique metric and torsion-free connection that is compatible with the complex structure.
- There are classes of $\mathcal{C}_\hbar - \mathcal{C}_{\hbar'}$-bimodules together with connections of constant curvature (only dependent on $\hbar, \hbar'$).