The signature problem in Noncommutative Geometry

Fabien Besnard
EPF
fabien.besnard@epf.fr

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Introduction

Geometry/Topology	Algebra
locally compact space	abelian C^*-algebra
compact	unital
measured space	abelian VN algebra
vector bundle	projective module of finite type
(riemannian) Spin manifold	real spectral triple

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Def: A (real, even) spectral triple is a multiplet $(\mathcal{A}, \mathcal{H}, \pi, D, J, \chi)$ with $\mathcal{A}$ a C^*-algebra, $\mathcal{H}$ a Hilbert space, π a rep. of $\mathcal{A}$, D, χ linear and J antilinear s.t.

1. $\chi^2 = 1$, $\chi^* = \chi$, $[\chi, \pi(\mathcal{A})] = 0$, $\{\chi, D\} = 0$,
2. $D^* = D$ + some analytical conditions
3. $J^2 = \epsilon$, $J^*J = 1$, $[D, J] = 0$, $J\chi = \epsilon''\chi J$.

Ex: riemannian spin manifold $M \longrightarrow$ canonical triple $\mathcal{A} = \mathcal{C}(M)$, $\mathcal{H}$ = square integrable spinor fields, $D = i\sum_k e_k \nabla^S_{e_k}$, J anticommutes with real vector fields.

Under some regularity conditions, there are essentially no other example with $\mathcal{A}$ commutative (Connes' reconstruction theorem[1]).

[1] J. Noncommut. Geom. 7 (2013) 1-82 arXiv:0810.2088

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Connes' distance formula

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$$d(\omega,\omega') = \sup_{a\in\mathcal{A}}\{|\omega(a)-\omega'(a)|,\ \|[D,a]\|\leq 1\}$$

(Connes' distance formula)

$\longrightarrow$ Gives back the geodesic distance in the case of a manifold.

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Remark on 1-forms:

$$\begin{tikzcd} 1-\text{form} \arrow[r, "\sharp"] \arrow[rrr, bend right, "\gamma"'] & \text{vector field} \arrow[r, "\subset"] & \text{section of } \mathbb{C}l(TM) \arrow[r, "\rho"] & \text{operator on spinor fields} \end{tikzcd}$$

Ex: If a is a smooth function, $[D, \pi(a)] = i\gamma(da)$.

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Ex: If a is a smooth function, $[D,\pi(a)] = i\gamma(da)$.

$$\Rightarrow \{\sum_i \pi(a_i)[D,\pi(b_i)]\} = \gamma(\Omega^1(M))$$

On a general ST:

$$\boxed{\Omega^1_D(\mathcal{A}) := \{\sum_i \pi(a_i)[D,\pi(b_i)]\} \text{ (noncommutative 1-forms)}}$$

2 points separated by a distance δ. Which spectral triple ?

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A discrete example

2 points separated by a distance δ. Which spectral triple ?

$$\rightarrow \mathcal{A} = \mathbb{C}^2, \mathcal{H} = \mathbb{C}^2, D = \frac{1}{\delta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, [D, \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}] = \frac{a(2)-a(1)}{\delta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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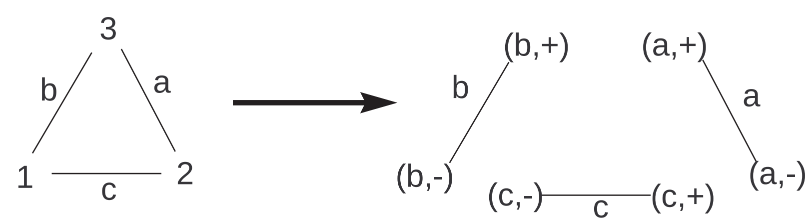

$b^+=a^+=3$

$(b,+) \neq (a,+)$

(split graph)

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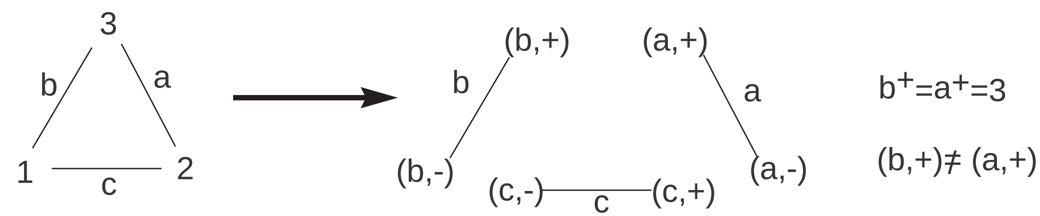

(split graph)

$G = (V, E)$ finite graph, $\delta : E \longrightarrow \mathbb{R}_+^*$ weight function

$\tilde{E} := E \times \{-;+\}$, $H = L^2(\tilde{E}) = \mathbb{C}^E \otimes \mathbb{C}^2$ + canonical $\langle ., . \rangle$.

$$\pi(a)F(e,\pm) = a(e^\pm)F(e,\pm) = \bigoplus_{e\in E}\begin{pmatrix} a(e^-) & 0 \\ 0 & a(e^+) \end{pmatrix}.$$

$$DF(e,\pm) = \frac{1}{\delta_e}F(e,\mp) = \bigoplus_{e\in E}\frac{1}{\delta_e}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\chi F(e,\pm) = \pm F(e,\pm), \; JF(e,\pm) = \overline{F(e,\pm)}$$

The signature question

Space and time oriented spin manifold M of signature $(p, q), q \neq 0$

$$\Big\Downarrow \text{canonical “triple”}$$

algebra of functions, ~~Hilbert~~ Krein space of spinor fields, D, J, χ

$$\Big\Downarrow \text{reconstruction}$$

$$M, g$$

The gang of four signs in semi-riemannian NCG

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$$M, \cancel{g}, (p,q)?$$

J and χ satisfy $J^2 = \epsilon$, $J\chi = \epsilon'' \chi J$, $J^\times = \kappa J$, $\chi^\times = \epsilon'' \kappa'' \chi$.

How far can you go with that ?

Convention: Clifford algebra of (V, g) generated by V with relations

$$vv' + v'v = 2g(v, v')$$

Canonical antiautomorphism: $(v_1 \dots v_k)^T = v_k \dots v_1$.

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$$\begin{tikzcd}
(V, g) \arrow[r, leftrightarrow] \arrow[d, dashed] & Cl(V, g) \arrow[r, leftrightarrow] & (\mathbb{C}l(V), c) \arrow[d, leftrightarrow] \\
(S, (.,.)^{+}_{S}) \arrow[r, leftrightarrow] & (\mathrm{End}(S), \times) \arrow[r, phantom, "\cong"] & (\mathbb{C}l(V), \times)
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up to $\lambda \neq 0$ $\qquad$ S a spinor module $\qquad$ $a^{\times} := c(a^T)$

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- The Robinson forms are characterized by $v^\times = v$, $v \in v$.
- There are also the anti-Robinson forms : $(\psi, \psi')^-_S := (\psi, \chi\psi')^+_S$, choosing $a^\times = Ad_\chi(c(a^T))$, and s.t. $v^\times = -v$, $v \in V$.

Fundamental symmetries, real structures, and the cardinal conventions

- There exist *fundamental symmetries* $\eta_\pm$ s.t. $\eta_\pm^2 = 1$ and $(., \eta_\pm .)^\pm \geq 0$.
- In case $p = 1$ (resp. $q = 1$), η_+ (resp. η_-) is just a normalized future-pointing vector (useful later !).
- There also exist two canonical real structures (up to a phase): J^+ s.t. $Ad_{J^+} = c$ and J^- s.t. $Ad_{J^-} = Ad_\chi \circ c$.

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In every case, we have $\chi^2 = 1, \eta^2 = 1$, $\chi\chi^{*_\eta} = 1$, $JJ^{*_\eta} = 1$ and

$$J^2 = \epsilon, J\chi = \epsilon''\chi J, J\eta = \epsilon\kappa\eta J, \eta\chi = \epsilon''\kappa''\chi\eta$$

with $\epsilon = (-1)^{\frac{n(n+2)}{8}}, \epsilon'' = (-1)^{n/2}, \kappa = (-1)^{\frac{m(m+2)}{8}}, \kappa'' = (-1)^{m/2}$ where m, n are defined mod 8 according to the cardinal conventions :

Convention	m	n	J	η
West-coast	$p+q$	$p-q$	J_-	η_+
East-coast	$p+q$	$q-p$	J_+	η_-
North-coast	$-p-q$	$p-q$	J_-	η_-
South-coast	$-p-q$	$q-p$	J_+	η_+

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Rem: If $(s,t) = (q,p)$ (West), $(s,t) = (p,q)$ (East), $(s,t) = (-q,-p)$ (North), $(s,t) = (-p,-q)$ (South), then $m = s+t$ and $n = t-s$.

Def: A mod 8 space-time representation is a quadruplet $(\mathcal{H}, J, \eta, \chi)$ with $\mathcal{H}$ a Hilbert space, χ and η linear endomorphisms and J antilinear s.t.

1. $\eta^2 = \chi^2 = 1, \chi^* = \chi, \eta^* = \eta,$
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We define (m, n) s.t.

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mod 8 space-time representations

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The space-time dimension couple (t, s) s.t. $m = s + t$ and $n = t - s$ is defined modulo $(4, 4)$.

$m \backslash n$	0	2	4	6
0	(0,0) (4,4)	(1,7) (5,3)	(2,6) (6,2)	(3,5) (7,1)
2	(1,1) (5,5)	(2,0) (6,4)	(3,7) (7,3)	(0,2) (4,6)
4	(2,2) (6,6)	(3,1) (7,5)	(4,0) (0,4)	(1,3) (5,7)
6	(3,3) (7,7)	(4,2) (0,6)	(5,1) (1,5)	(6,0) (2,4)

- There are four embeddings of category $Cliff \hookrightarrow Rep$.
- We know that $Cl(p_1, q_1)\hat{\otimes}Cl(p_2, q_2) \simeq Cl(p_1 + p_2, q_1 + q_2)$.
- We define the graded tensor product of mod 8 st rep such that it is preserved by the above embeddings.

Tensor products

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- We define the graded tensor product of mod 8 st rep such that it is preserved by the above embeddings.
- The formula is: $(\mathcal{H}, J, \eta, \chi) = (\mathcal{H}_1, J_1, \eta_1, \chi_1)\hat{\otimes}(\mathcal{H}_2, J_2, \eta_2, \chi_2)$ with

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2, \\
J &= J_1\chi_1^{|J_2|}\hat{\otimes}J_2\chi_2^{|J_1|}, \\
\eta &= i^{|\eta_1||\eta_2|}\eta_1\chi_1^{|\eta_2|}\hat{\otimes}\eta_2\chi_2^{|\eta_1|}, \\
\chi &= \chi_1\hat{\otimes}\chi_2.
\end{aligned}$$

Formula for η in terms of Krein products:

$$(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2) = i^{|\eta_1||\eta_2|}(\phi_1, \psi_1)_1(\phi_2, \chi_2^{|\eta_1|}\psi_2)_2.$$

(Recall that: $(T_1\hat{\otimes}T_2)(\psi_1 \otimes \psi_2) = (-1)^{|T_2||\psi_1|}T_1\psi_1 \otimes T_2\psi_2$ is the same operator as $T_1\chi_1^{|T_2|} \otimes T_2$)

The Lorentzian case

The general framework

First paper: A. Strohmaier, *On Noncommutative and Pseudo-Riemannian Geometry*, J. Geom. Phys. 56(2) (2006) math-ph/0110001
Defines an even *semi-Riemannian spectral triple* to be:

1. A $*$-algebra $\mathcal{A}$ and a Krein space $(K, (.,.))$.
2. A faithful representation π s.t. $\pi(a^*) = \pi(a)^\times$.
3. A grading χ such that $\chi^2 = 1$, $\chi^\times = \pm\chi$ and $[\chi, \pi(\mathcal{A})] = 0$.
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1. Order 0 condition: $[\pi(a), J\pi(b)^\times J^{-1}] = 0$ for all $a, b \in \mathcal{A}$,
2. Order 1 condition: $[[D, \pi(a)], J\pi(b)^\times J^{-1}] = 0$.

What is specifically Lorentzian ?

1. M. Paschke, A. Sitarz, *Equivariant Lorentzian Spectral Triples*, math-ph/0611029

$$\eta = i \sum_j J\pi(a_j^0)J^{-1}\pi(a_j)[D,\pi(b_j)], a_j^0, a_j, b_j \in \mathcal{A}$$

2. N. Franco, M. Eckstein, *An algebraic formulation of causality for noncommutative geometry*, Class. Quantum Grav. 30 (13) (2013).

$$\eta = -N[D,T], N \in \pi(\mathcal{A}), T \in End(\mathcal{H})$$

3. Causal approach. C^*-algebra $\longrightarrow$ I^*-algebra. F. Besnard, *Noncommutative ordered spaces, examples and counterexamples*, Class. Quantum Grav. 32 (2015) arXiv:1312.2442

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We will follow an approach close to 1, with some differences:

1. η not normalized, not unique, no a_j^0.
2. No C^*-algebra structure fixed in advance.

An (even, real) spectral spacetime is a multiplet $S = (\mathcal{A}, K, (.,.), \pi, D, J, \chi)$, where:

1. $(K, (.,.))$ is a Krein space,
2. $\mathcal{A}$ is an algebra and π is a faithful representation on K,
3. D is an operator on K such that $D^\times = D$,
4. $\chi \in B(K)$ is such that $\chi^2 = 1$, $[\pi(a), \chi] = 0$ for all $a \in A$, $\chi D = -D\chi$ and $\chi^\times = -\chi$,
5. J is an antilinear operator on K such that $J^2 = \epsilon$, $[J, D] = 0$, $J\chi = \epsilon''\chi J$, $J^\times J = -1$.
6. $\exists \beta \in \Omega^1_D(\mathcal{A}, \pi)$, a "time-orientation form" s.t. $\beta^\times = \beta$, $J\beta = -\beta J$, $\langle ., . \rangle_\beta := (., \beta^{-1}.)$ is positive definite.

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Problem: no C^*-structure $\Rightarrow$ no space !

The Hilbert adjoint defined by β is $A^{*_\beta} := \beta A^\times \beta^{-1}$.

Def: A spectral spacetime S is reconstructible if $\exists\beta$ s.t. $\pi(\mathcal{A})^{*_\beta} = \pi(\mathcal{A})$.

Different “observers”, same space

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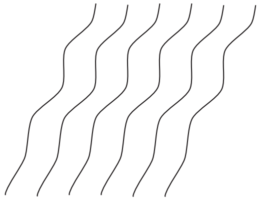
Manifold case: $\beta \longleftrightarrow$ congruence of observers.



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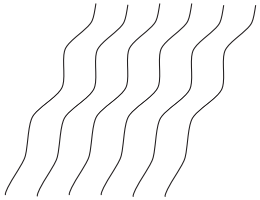

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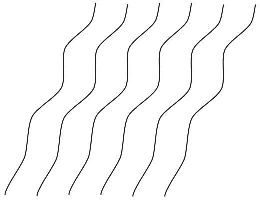

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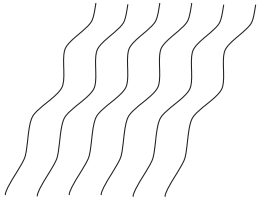

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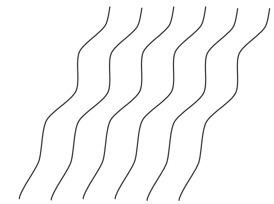

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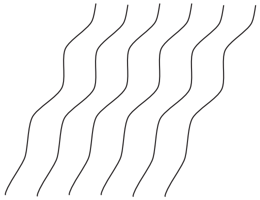

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☞ the structure/state spaces are isomorphic: "Two observers see the same space"

☞ order 0 + order 1 conditions + reconstructibility $\Rightarrow [\beta, \pi(\mathcal{A})] = 0$ and $\pi(a)^{\times} = \pi(a)^{*_\beta}$.

Def: A Jordan-Banach algebra is a Jordan algebra A over $\mathbb{R}$ with a complete norm s.t.

$$\|a \circ b\| \leq \|a\|\|b\|, \|a^2\| = \|a\|^2, \|a^2\| \leq \|a^2 + b^2\|$$

Universal observables

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Let S be a SST and let β be a time-orientation form. We define:

1. The JB algebra of β-observables $U^\beta(S) := \{a \in \mathcal{A} | \pi(a)^{*_\beta} = \pi(a)\}$,
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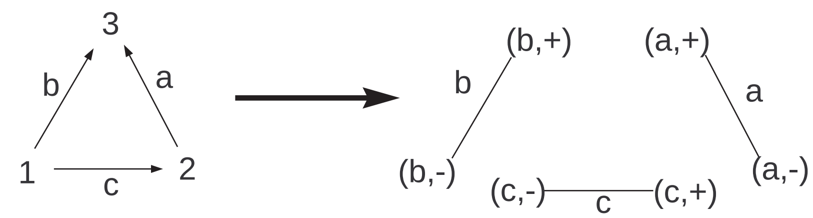
One can prove that:

- $U_J(S)$ is always associative, hence $\approx \mathcal{C}(X, \mathbb{R})$ for some space X.
- If the order 0 and 1 conditions are satisfied, then $U(S) = \{a \in \mathcal{A} | \pi(a)^\times = \pi(a)\}$.

Examples

The split graph again

$G = (V, E)$ finite graph with weight function $\delta : E \to \mathbb{R}_+^*$ + an orientation



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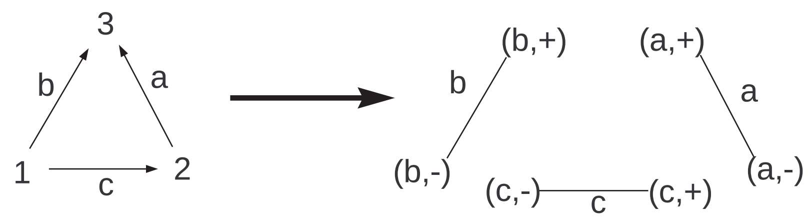

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Same algebra, same rep. on same vector space K, but with Krein product

$$(F,G)_\omega = i\sum_{e\in E}\left(F(e,-)\overline{G(e,+)} - F(e,+)\overline{G(e,-)}\right),$$

$$D=\bigoplus_{e\in E}\frac{-i}{\delta_e}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \text{ and } JF(e,\pm)=\pm\overline{F(e,\mp)}.$$

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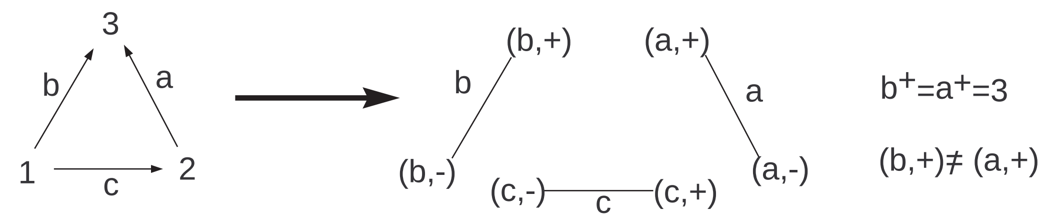

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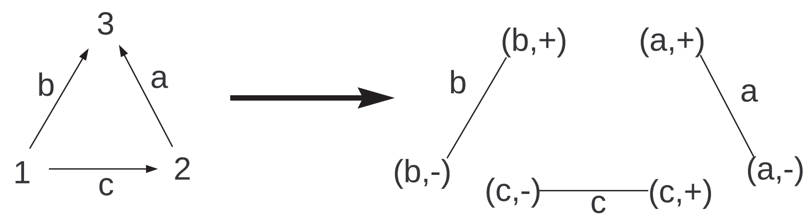

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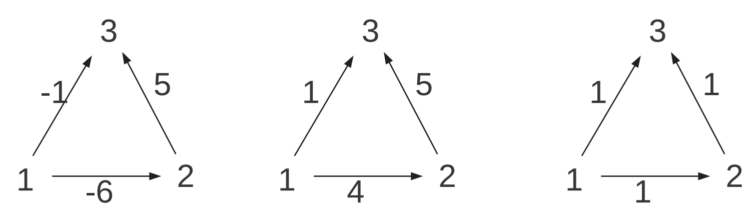

real self-adjoint 1-form (exact)

orientation 1-form

normalized orientation 1-form

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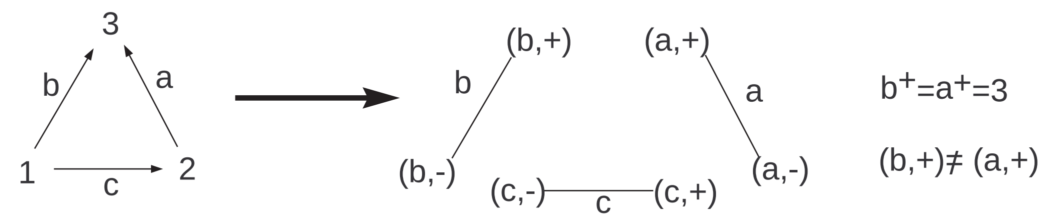

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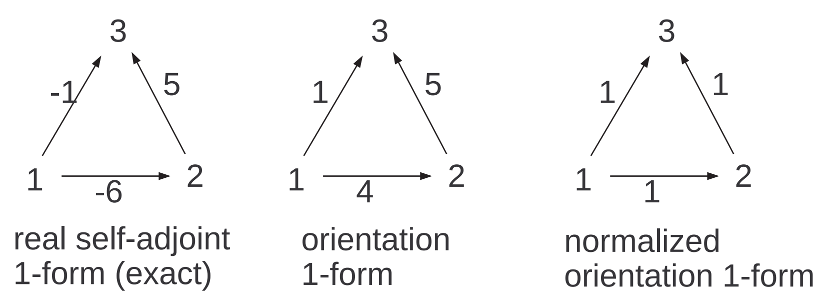

$\longrightarrow$ It is stably causal iff the graph has no directed loop.

A noncommutative generalization: the split Dirac structure

- $G = (V, E, s, t, \delta)$ is p.w.d. and connected, $\tilde{E}$ its split graph.
- For each $v \in V$, (X_v, g_v) is n-dimensional antilorentzian.
- $\mathcal{A}$ is the algebra of sections $a : v \mapsto a_v \in \mathbb{C}l(X_v, g_v)^0$
- S_v spinor representation of $\mathbb{C}l(X_v, g_v)$, $(.,.)_v$ compatible Krein product, C_v canonical real structure, $(h_e)_{e\in E} : S_{e^-} \longrightarrow S_{e^+}$ discrete connection.
- K space of sections $F : (e, \pm) \mapsto F(e, \pm) \in S_{e^\pm}$ of the discrete spinor bundle $\bigcup_{(e,\pm)\in\tilde{E}} S_{e^\pm}$ over $\tilde{E}$.

$$(F, G) = \sum_{e\in E} \big((F(e,+), h_e^+ G(e,-)) + (F(e,-), h_e^- G(e,+))\big)$$

- Representation of $\mathcal{A}$ on K: $(\pi(a)F)(e, \pm) = a(e^\pm) \cdot F(e, \pm)$.
- Dirac: $D = \bigoplus_{e\in E} \frac{i}{\delta_e} \begin{pmatrix} 0 & \gamma_e^+ h_e^+ \\ \gamma_e^- h_e^- & 0 \end{pmatrix}$ with $\gamma_e^\pm \in X_{e^\pm}$ ("gamma matrices").
- $(\chi F)(e, \pm) = \chi_{e^\pm} F(e, \pm)$, $(JF)(e, \pm) = h_e^\pm J_{e^\mp} F(e, \mp)$

The SDS as a SST (1)

We say that the discrete connection is

- metric if the h_e are Krein-unitary,
- preserves spin if
$$\begin{tikzcd} S_{e-} \arrow[r, "J_{e-}"] \arrow[d, "h_e^+"'] & S_{e-} \arrow[d, "h_e^+"] \\ S_{e+} \arrow[r, "J_{e+}"] & S_{e+} \end{tikzcd}$$
commutes,
- is Clifford iff for all e there exists a linear map $\Lambda_e : X_{e-} \to X_{e+}$ such that the following diagram commutes for all $v \in X_{e-}$:
$$\begin{tikzcd} S_{e-} \arrow[r, "v"] \arrow[d, "h_e^+"'] & S_{e-} \arrow[d, "h_e^+"] \\ S_{e+} \arrow[r, "\Lambda_e(v)"] & S_{e+} \end{tikzcd}$$
- preserves orientation if $\chi_{e^\pm} h_e^\pm = h_e^\pm \chi_{e^\mp}$.

Th: The split Dirac structure is a SST if and only if

1. The elements $\gamma_e^{\pm}$ satisfy

$$\gamma_e^{+} = h_e^{+} \gamma_e^{-} h_e^{-}, \forall e \in E$$

2. The discrete connection is metric, spin and orientation preserving.

3. If the $\gamma_{e^{\pm}}$ span $X_{e^{\pm}}$ then the discrete connection is Clifford by (1).

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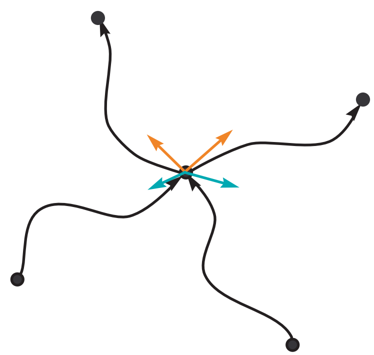
Th: When $n > 2$ and $(h_e)_{e \in E}$ is Clifford, the following are equivalent:

- the SDS is recontructible,
- there exists a future-directed “covariantly constant vector field”, i.e. $v \mapsto u_v \in X_v^+$ s.t. $u_w = H_{v \rightsquigarrow w}(u_v)$, for any path $v \rightsquigarrow w$,
- the discrete holonomy group fixes a timelike direction.

SDS and the discretization of the Dirac operator

$G =$ embedded graph, $X_v = T_v M$, $S =$ discrete spinor bundle. Discretization of the Dirac[2]

$$(\tilde{D}\psi)(v) = i \sum_{e|e^+=v} \frac{1}{2l_e}\gamma_e h_e(\nabla^S)\psi(e_-) + i \sum_{e|e_-=v} \frac{1}{2l_{\bar{e}}}\gamma_{\bar{e}} h_{\bar{e}}(\nabla^S)\psi(e^+)$$



$$i : \Gamma(S) \to K,$$
$$\phi(e, \pm) = \phi(e^{\pm})$$
$$\Pi : K \to \Gamma(S),$$
$$\Pi F(v) = \frac{1}{d_v}\Big(\sum_{e|t(e)=v} F(e,+) + \sum_{e|s(e)=v} F(e,-)\Big)$$

Prop If the SDS is suitably defined and G is made of orthonormal geodesic segments then $\Pi \circ D \circ i = \frac{2}{i d_v}\tilde{D}$ and the SDS is automatically a SST.

[2] M. Marcolli, W. D. van Suijlekom, *Gauge Networks in noncommutative geometry*, J. Geom. Phys. **75**, 71–91 (2014), arXiv:1301.3480

Conclusion

Much remains to be done !

1. Other signatures ?
2. Analytical conditions ?
3. Odd dimensional case ?
4. Classification of finite dimensional SST ?
5. Stable causality for SDS ?
6. Updating of isocone theory.

References:

- On space/time dimension couples: N. Bizi, C. Brouder, FB, abs/1611.07062
- On SST: N. Bizi, FB, *On the definition of spacetimes in Noncommutative Geometry, Part I*, J. Geom. Phys. abs/1611.07830
- FB, *On the definition...*, Part II, abs/1611.07842

- SST $\longrightarrow$ ST: Let $S = (\mathcal{A}, K, (.,.), \pi, D, C, \chi)$ be a SST of KO-dim n mod 8, and β be a positive time orientation 1-form. Define $C_\beta = \beta C$, $\chi_\beta = -\chi$, $D_\beta = \frac{1-i}{2} D + \frac{1+i}{2} \beta D \beta^{-1}$, and $a^{*_\beta} = \pi^{-1}(\pi(a)^{*_\beta})$.

  Then $S_\beta = ((\mathcal{A}, *_\beta), K, \langle .,. \rangle_\beta, \pi, D_\beta, C_\beta, \chi_\beta)$ is a real even spectral triple with KO-dimension $2 - n$ if and only if $\beta^2 = 1$.

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- ST $\longrightarrow$ SST: Let $S = (\mathcal{A}, H, \pi, D, C, \chi)$ be a real even ST of KO-dim n mod 8. Let $\omega \in \Omega^1_D(\mathcal{A}, \pi)$ be s.t. $\omega = \omega^*$ and let

$$D_\omega = \frac{1-i}{2} D + \frac{1+i}{2} \omega D \omega^{-1}; (.,.)_\omega = \langle ., \omega . \rangle, C_\omega = \omega C, \chi_\omega = -\chi$$

  Then $S_\omega = (\mathcal{A}, H, \pi, D_\omega, C_\omega, \chi_\omega)$ is a SST of KO-dim $2 - n$ mod 8 on which ω is a positive t.o. 1-form if and only if

$$[C, \omega] = 0, \ \omega^2 = 1, \text{ and } \omega \in \Omega^1_{D_\omega}(\mathcal{A}, \pi)$$

A non-reconstructible SST

Example of non-reconstructible SDS:

$$\begin{tikzcd}
 & 3 \arrow[dl, "c"', "\mathrm{Id}"] & \\
1 \arrow[rr, "a", "\mathrm{boost}"'] & & 2 \arrow[ul, "b"', "\mathrm{Id}"]
\end{tikzcd}$$