The Standard Model in Noncommutative Geometry: fermions as internal forms

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Goal

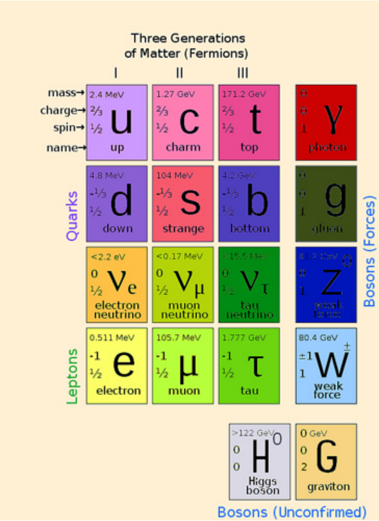
Unveil the geometric nature of the multiplet of fundamental fermions in the Standard Model of fundamental particles

Plan

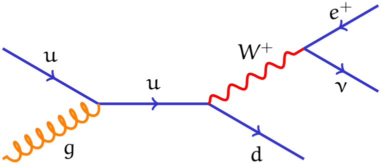
1. *few words on the SM and its nocommutative geometric formulation νSM*
2. *concept of quantum Dirac spinors and quantum de Rham forms as Morita equivalence bimodules of the algebra of sections of the quantum analogue of Clifford bundle*
3. *application to νSM.*

Proviso: *quantum = noncommutative (NC)*

(Unreasonably) successful Standard Model



& interactions



governed by:

Lagrangian

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu\partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu\partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu\partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu\partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)$$

$$- Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)) - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu$$

$$+ \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2s_wc_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H\partial_\mu H$$

$$- 2M^2\alpha_h H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - \frac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \beta_h\left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h - g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \frac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2\right.$$

$$\left.+4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) - gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H)\right.$$

$$\left.+W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + M(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+)$$

$$- ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right)$$

$$- \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^-$$

$$- g^2s_w^2A_\mu A_\mu\phi^+\phi^- + \frac{1}{2}ig_s\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial + m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_wA_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right)$$

$$+ \frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{d}^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2 - 1 - \gamma^5)d^\lambda_j) + (\bar{u}^\lambda_j\gamma^\mu(1 - \frac{8}{3}s_w^2 + \gamma^5)u^\lambda_j)\} + \frac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5){U^{lep}}_{\lambda\kappa}e^\kappa) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right)$$

$$+ \frac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_e(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa}(1+\gamma^5)e^\kappa\right) + \frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_e(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1+\gamma^5)\nu^\kappa)\right.$$

$$\left.-m^\kappa_\nu(\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right) - \frac{g}{2}\frac{m^\lambda_\nu}{M}H(\bar{\nu}^\lambda\nu^\lambda) - \frac{g}{2}\frac{m^\lambda_e}{M}H(\bar{e}^\lambda e^\lambda) + \frac{ig}{2}\frac{m^\lambda_\nu}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda) - \frac{ig}{2}\frac{m^\lambda_e}{M}\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda) - \frac{1}{4}\bar{\nu}_\lambda M^R_{\lambda\kappa}(1-\gamma_5)\hat{\nu}_\kappa - \frac{1}{4}\overline{\bar{\nu}_\lambda M^R_{\lambda\kappa}(1-\gamma_5)\hat{\nu}_\kappa}$$

$$+ \frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right) + \frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right) - \frac{g}{2}\frac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j)$$

$$+ \frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)$$

Conceptually/Geometrically:

$\textcircled{\Phi}$ $U(1) \times SU(2) \times SU(3)$ *gauge fields (bosons)*
minimally coupled to matter fields (fermions) & Higgs field (boson)
$+$ *2nd quantization with gauge fixing, spontaneous symmetry breaking, regularization & perturbative renormalization*

However unexplained:
- (though constrained) contents of particles (especially 3 families)
- several parameters,
- the fourth known interaction not included:

gravitation

with its own fundamental symmetry:

general relativity (diffeomorphisms)

There have been various attempts to settle some of the above:
GUT based on a simple group $SU(5)$ or $SO(10)$,
modern variants of Kaluza-Klein model with 'compactified' internal dimensions, and other more recent and fashionable ... ↩

νSM

Another one, νSM, has been formulated in the framework of noncommutative geometry by A. Connes et.al.

Rather than **groups** in the usual "commutative" geometry of SM:

Ⓜ $U(1) \times SU(2) \times SU(3)$ *connection (~ multiplet of vectors) on (a multiplet of) spinors; & a doublet of scalars*

νSM is primarily based on **algebras**.

It adds to the 75 years-old Gelfand-Naimark (anti)equivalence:

$$\boxed{topological\ spaces \longleftrightarrow commutative\ C^* - algebras}$$

and to the Serre-Swan equivalence:

$$\boxed{vector\ bundles \longleftrightarrow modules}$$

other data to encode *smoothness*, *calculus* and *metric* on M.

Intro

The first datum is a Hilbert space H that carries a unitary representation of a (possibly noncommutative) $*$-algebra A (thus also of its norm closure C^*-algebra).

The second one is an analogue of Dirac operator $D = D^\dagger$ on H.

Together with the algebra A they satisfy some analytic conditions,

$$[D, A] \subset \mathcal{B}(H), \quad (D - z)^{-1} \in \mathcal{K}(H), \ \ z \notin spec(D)$$

so that they form a **spectral triple** (S.T.) $\hookleftarrow$

$$(A, H, D).$$

A S.T. is called *even* if $\exists$ a $\mathbb{Z}_2$-grading χ of H, $\chi^2 = \chi^\dagger = \chi$, s.t.

$$[\chi, A] = 0, \quad \{\chi, D\} = 0.$$

Intro 2

A S.T. is called *real* if $\exists$ a real structure, i.e. antiunitary J on H, s.t.
denoting B' the commutant of $B \subset \mathcal{B}(H)$,

$$JAJ^{-1} \subset A', \quad \text{(order 0 condition)}. \tag{1}$$

In addition, we call

$$JAJ^{-1} \subset [\not{D}, A]', \quad \text{(order 1 condition)} \tag{2}$$

and

$$J[\not{D}, A]J^{-1} \subset [\not{D}, A]', \quad \text{(order 2 condition)}. \tag{3}$$

Names: A-bimodule spanned by $[D, A]$ - 1-forms
A-algebra generated by $[D, A]$ - forms (with derivative $[D, \cdot]$)
or, quite as in [Lord et.al.] - Clifford algebra denoted $Cl_D(A)$.

This permits right actions

$$\triangleleft\, b := Jb^*J^{-1} \text{ on } H,$$

so that the order 0, 1, 2 conditions mean that H is a

$$A - A, \quad A - Cl_D(A) \quad \text{and} \quad Cl_D(A) - Cl_D(A)$$

bimodule, respectively.

Intro 3

Further A. Connes formulated few other important properties (which permit to reconstruct the geometrical data); one of them is the request that

$$J^2 = \epsilon\,\mathrm{id}_H, \quad JD = \epsilon' DJ, \quad J\chi = \epsilon''\chi J \quad \text{(in the even case)} \tag{4}$$

for some $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$, that specify the "KO-dimension" mod. 8.

Canonical S.T.

The prototype example is the *canonical* S.T. on a spin manifold M

$$(C^\infty(M), L^2(S), D\!\!\!/),$$

where $C^\infty(M)$ is the algebra of smooth complex functions on M,
S is the $\text{rank}_{\mathbb{C}} = 2^{[n/2]}$ bundle of Dirac spinors on M,
whose sections carry a faithful irrep γ of the algebra of sections of the (simple part of) Clifford bundle $\mathbb{C}l(M)$

$$\gamma : \Gamma(\mathbb{C}l(M)) \underset{\approx}{\to} \mathrm{End}_{C^\infty(M)}\Gamma(S) \approx \Gamma(S) \otimes_{C^\infty(M)} \Gamma(S)^* \qquad (5)$$

and $D\!\!\!/$ is the usual Dirac operator on M:

$$D\!\!\!/ = \gamma \circ \nabla = \sum_j^n \gamma^j \nabla_j \quad \text{(locally)}. \qquad (6)$$

(!) Note that (5) means that (after norm completion)

$$\boxed{\Gamma(S)\,\text{is a Morita equivalence}\,\Gamma(\mathbb{C}l(M)) - C^\infty(M)\,\text{bimodule}} \qquad (7)$$

and this exactly characterizes Spin_c manifolds M [Plymen].

Canonical S.T. 2

Note also that since $[\not{D}, a] = \gamma(da)$ indeed we have

$$\mathbb{C}l_{\not{D}}(C^\infty(M)) \approx \Gamma(\mathbb{C}l(M)).$$

Next if dim$M$ is even, $\exists$ a "chiral" $\mathbb{Z}_2$-grading χ_S of $L^2(S)$.

Furthermore $\exists$ a real structure J_S (given by 'charge conjugation'), that satisfies order 0 and 1 condition, and obviously <u>not</u> the order 2 condition since it implements the Morita equivalence (12). Such J_S, together with (12) provides precisely the algebraic characterization of spin manifolds.

We also mention that the signs $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$ associated to $\not{D}, \chi_S, J_S$ correspond to the "KO-dimension" equal n mod. 8.

Finally, the canonical S.T. fully encodes the geometric data on M, that can be indeed reconstructed [Connes]. ↩

de Rham-Hodge S.T.

But it is not the only natural S.T.; on an oriented Riemannian manifold there is also

$$(C^\infty(M), L^2(\Omega(M)), d + d^*),$$

where $\Omega(M)$ is the space of de Rham differential forms on M ↩
with the hermitian form induced by the metric g on M,
d is the exterior derivative and d^* its adjoint.

The operator $d + d^*$ is Dirac-type:

$$d + d^* = \lambda \circ \nabla, \tag{8}$$

where the representation

$$\lambda : \Gamma(\mathbb{C}l(M)) \to \mathrm{End}_{C^\infty(M)}\Omega(M), \;\; \lambda(v) = v \wedge - v \lrcorner, \; v \in T^*M \tag{9}$$

is equivalent to the left regular self-representation of $\Gamma(\mathbb{C}l(M))$.

Clearly $[d + d^*, a] = \lambda(da)$ so we again have

$$\mathbb{C}l_{d+d^*}(C^\infty(M)) \approx \Gamma(\mathbb{C}l(M)).$$

There is also an anti-representation

$$\rho : \Gamma(\mathbb{C}l(M)) \to \mathrm{End}_{C^\infty(M)}\Omega(M), \;\; \rho(v) = (v \wedge + v \lrcorner)\chi_\Omega, \; v \in T^*M, \tag{10}$$

where

$$\chi_\Omega = \pm 1 \tag{11}$$

on even forms $\Omega(M)^{even}$, respectively odd forms $\Omega(M)^{odd}$, that is equivalent to the right regular self anti-representation of $\Gamma(\mathbb{C}l(M))$.

Furthermore, since λ_v and $\rho_{v'}$ commute, $\Omega(M)$ is a $\Gamma(\mathbb{C}l(M))$-$\Gamma(\mathbb{C}l(M))$ bimodule, which is equivalent to $\Gamma(\mathbb{C}l(M))$.

Thus (after norm completion)

$$\boxed{\Omega(M) \,\mathrm{is\,a\,Morita\,equivalence}\,\Gamma(\mathbb{C}l(M)) - \Gamma(\mathbb{C}l(M))\mathrm{bimodule}} \tag{12}$$

which characterizes $\Omega(M)$ up to $\otimes$ with a complex line bundle.

de Rham-Hodge S.T. 3

Besides the grading χ_Ω by parity which $\exists$ on any M, if dim$M = n = 2m$ is even $\exists$ another grading given by the normalized Hodge operator

$$\chi'_\Omega := i^{k(n-k)+m} * : \Omega^k(M) \to \Omega^{n-k}(M) \tag{13}$$

On any M $\exists$ also a real structure on $\Omega(M)$

$$J_\Omega := c.c,$$

which satisfies the order 0 and 1 conditions but not order 2, and so can not implement the $\Gamma(\mathbb{C}l(M))$-$\Gamma(\mathbb{C}l(M))$ self-Morita equivalence.

For that we need another J'_Ω on $\Omega(M)$ that interchanges the actions λ and ρ. It turns out that there is one:

$$J'_\Omega(e_{j_1} \wedge \cdots \wedge e_{j_k}) = e_{j_k} \wedge \cdots \wedge e_{j_1}, \quad 0 \leq k \leq n, \tag{14}$$

which corresponds to the main anti-involution on $\Gamma(\mathbb{C}l(M))$ and can be simply written on $\Omega^k(M)$ as

$$J'_\Omega = (-)^{k(k-1)/2} \circ c.c. \tag{15}$$

It satisfies all the order 0, 1 and 2 conditions and does implement the $\Gamma(\mathbb{C}l(M)$-$\Gamma(\mathbb{C}l(M))$ self-Morita equivalence (!).

de Rham-Hodge S.T. 5

We mention that for $d + d^*$, and respectively:

- χ_Ω, J_Ω one has $\epsilon = 1, \epsilon' = 1, \epsilon'' = 1$ and so KO-dim=0;
- χ'_Ω, J_Ω one has $\epsilon = 1, \epsilon' = 1, \epsilon'' = (-1)^m$ and so KO-dim=0 if n=0 mod 4, and 6 if n=2 mod 4 [Rubin, M. Thesis];
- χ_Ω, J'_Ω one has $\epsilon = 1, \epsilon' = 1, \epsilon'' = 1$ and so KO-dim=0;
- χ'_Ω, J'_Ω one has $\epsilon = 1, \epsilon' = 1$, but on k-forms

$$J'_\Omega \chi'_\Omega = (-)^k \chi'_\Omega J'_\Omega \tag{16}$$

so ϵ'' is not just a sign but a grading (given by χ_Ω), which requires a generalization of the notion of KO-dimension.

Actually it is not known if, and with which additional conditions, the de Rham-Hodge S.T. equipped with any choice of χ's and J's as above may faithfully encode the geometric data on M, that can be then reconstructed.

νSM: A_F

The underlying arena of νSM [Connes, Chammseddine,...] is

ordinary (spin) manifold M $\times$ a finite quantum space F,

described by the algebra $C^\infty(M) \otimes A_F$, where

$$\boxed{A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).}$$

The Hilbert space is

$$L^2(S) \otimes H_F,$$

where

$$\boxed{H_F = \mathbb{C}^{96} =: H_f \otimes \mathbb{C}^3,}$$

with $\mathbb{C}^3$ corresponding to 3 generations, and

νSM: H_f

$$H_f = \mathbb{C}^{32} \simeq M_{8\times 4}(\mathbb{C})$$

with basis labelled by particles and antiparticles, we arrange as

$$\begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \\ \bar{\nu}_R & \bar{e}_R & \bar{\nu}_L & \bar{e}_L \\ \bar{u}_R^1 & \bar{d}_R^1 & \bar{u}_L^1 & \bar{d}_L^1 \\ \bar{u}_R^2 & \bar{d}_R^2 & \bar{u}_L^2 & \bar{d}_L^2 \\ \bar{u}_R^3 & \bar{d}_R^3 & \bar{u}_L^3 & \bar{d}_L^3 \end{bmatrix}$$

(1,2,3=colors).

νSM: π_F

The representation of A_F, diagonal in generations, on H_f is:

$$\pi_F(\lambda, q, m) = \begin{bmatrix} \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{q}} \\ 0 & 0 & & \end{array}\right] & 0_4 \\ 0_4 & \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array}\right] \end{bmatrix} \qquad (17)$$

Note that $\pi_F(A_F)$ is a real $*$-algebra of operators, and to get its complexification just replace $\bar{\lambda}$ by an independent $\lambda' \in \mathbb{C}$, and take $q \in M_2(\mathbb{C})$.

νSM: γ_F & J_F

The grading (the chirality operator) is $\gamma_M \otimes \gamma_F$,
where γ_F (diagonal in generations) on H_f reads:

$$\gamma_F = \begin{bmatrix} 1_2 & & \\ & -1_2 & \\ & & 0_4 \end{bmatrix} \otimes 1_4 + \begin{bmatrix} 0_4 & \\ & -1_4 \end{bmatrix} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} . \qquad (18)$$

The real conjugation is $J_M \otimes J_F$, where J_F on H_f is

$$J_F \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2^* \\ v_1^* \end{bmatrix} \qquad (19)$$

that satisfies the order 0 and 1 conditions (as in the classical case).

νSM: D_F

Finally, the Dirac operator is $D = \not{D}_M \otimes \mathrm{id} + \gamma_M \otimes D_F$, where Chamseddine-Connes':

$$
D_F = \left[\begin{array}{cccc|cccc}
 & & \bar{\Upsilon}_\nu & & \bar{\Upsilon}_R & & & \\
 & & & \bar{\Upsilon}_e & & & & \\
\Upsilon_\nu & & & & & & & \\
 & \Upsilon_e & & & & & & \\
\hline
\Upsilon_R & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}\right] \otimes \; e_{11} + \left[\begin{array}{cccc|cccc}
 & & \bar{\Upsilon}_u & & & & & \\
 & & & \bar{\Upsilon}_d & & & & \\
\Upsilon_u & & & & & & & \\
 & \Upsilon_d & & & & & & \\
\hline
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}\right] \otimes \; e_{11}^{\perp} +
$$

$$
e_{55} \otimes \left[\begin{array}{cc|cc}
0 & 0 & \bar{\Upsilon}_\nu & 0 \\
0 & 0 & 0 & \bar{\Upsilon}_e \\
\hline
\Upsilon_\nu & 0 & 0 & 0 \\
0 & \Upsilon_e & 0 & 0
\end{array}\right] + (e_{66} + e_{77} + e_{88}) \otimes \left[\begin{array}{cc|cc}
0 & 0 & \bar{\Upsilon}_u & 0 \\
0 & 0 & 0 & \bar{\Upsilon}_d \\
\hline
\Upsilon_u & 0 & 0 & 0 \\
0 & \Upsilon_d & 0 & 0
\end{array}\right],
$$

where Υ's are in $\mathrm{Mat}(3, \mathbb{C})$, or just in $\mathbb{C}$ for one generation.

νSM

With all that:

• $\mathcal{G} := \{U = uJuJ^{-1} \,|\, u \in A, det\,U = 1\} \simeq U(1)\times SU(2)\times SU(3)$ (S.M. gauge group)

• all the fundamental fermions in H have the correct S.M. charges w.r.t. $\mathcal{G}$ (broken to $U(1)_{em} \times SU(3)$)

• the 1-forms $a[D, b]$, $a, b \in A$ yield the S.M. gauge fields $A_\mu, W^\pm, Z, G_\mu$ (from the part D_M of D), plus the Higgs complex scalar (weak doublet) Higgs field (from the part D_F of D).

MERITS:

• gauge & Higgs field as a connection,

• explains why only the fundamental reps of $\mathcal{G}$,

• a simple spectral action Tr$f(D/\Lambda)$ reproduces the bosonic part of $\mathcal{L}_{SM}$ as the lowest terms of asymptotic expansion in Λ, & and $< \phi, D\phi >$ the (Wick-rotated) fermionic part

• couples to gravity on M

• Connes&Chamseddine claim to predict a new relation among the parameters of S.M.

Geometry of νSM

The above "almost commutative" geometry is described by a S.T.

$$(C^\infty(M), L^2(S), \not{D}) \times (A_F, H_f, D_F),$$

that is mathematically a product of the "external" canonical S.T. on spin manifold M with the "internal" *finite* S.T.

What is its geometric interpretation of (A_F, H_f, D_F) **?**

Does it also correspond to a (noncommutative) spin manifold ?
Are the elements of H_f "spinors" in some sense ?
In particular "Dirac spinors" ?

Or does it correspond rather to de-Rham forms ?

Or else ?

Dirac spinors: quantum

To answer this question, basing on a deeper understanding of the classical case, to accomplish the noncommutative case we define

Def (c.f. FD'A, LD)

An even spectral triple (A, H, D, χ) *is called* spin$_c$
if H *is a Morita equivalence* $\mathbb{C}l_D(A)$-A *bimodule*
(i.e. after norm-completion the algebras $\mathbb{C}l_D(A)$ *&* A *are maximal one w.r.t. the other),*
and it is called spin *if the right action of* $a \in A$ *is* Ja^*J^{-1}
(implemented by a real structure J *satisfying the 1st O.C.).*
Furthermore the elements of H *are called quantum Dirac spinors (sometimes named "charged" or "neutral", respectively).*

νSM: 1st result

Is the internal S.T. of νS.M. *spin*?
(like the external one = the canonical S.T. on M)

Bulding on and extending the classifications of [Krajewski] and [Paschke,Sitarz] the answer [FD'A, LD] is: **'NO'**
In fact, after a tedious and sistematic search we constructed

$$X = e_{55} \otimes (1 - e_{11}), \text{ s.t. } X \in (A)' \text{ but } X \notin JAJ.$$

(Can be evaded with a different grading and two extra $\neq 0$ matrix elements in D_F - the status of which is however under study since though desirable for the correct renormalized Higgs mass, they would have unobserved couplings to fermions).

But then, without such additions, may be the internal S.T. of νSM is rather an analogue of the other natural classical spectral triple, namely de-Rham forms?

de Rham forms: quantum

To answer this question, basing on a deeper understanding of the classical case of $\Omega(M)$ with χ_Ω and J'_Ω, we define

Def (c.f. FD'A, LD, AS)

An even spectral triple (A, H, D, χ) is called complex Hodge if H is a Morita selfequivalence $\mathcal{C}\ell_D(A)$-$Cl_D(A)$ bimodule, and Hodge if the right $\mathcal{C}\ell_D(A)$-action is implemented by J satisfying the 2nd O.C.).
Furthermore we then say that H consists of quantum complex or real de-Rham forms, respectively.

Theorem (LD, FD'A, AS)

For the internal spectral triple of the νS.M. with one generation the Hodge property holds whenever $\Upsilon_x \neq 0,\ \forall\, x \in \{\nu, e, u, d\}$ and

$$|\Upsilon_\nu| \neq |\Upsilon_u| \qquad \text{or} \qquad |\Upsilon_e| \neq |\Upsilon_d| \,.$$

About the proof: A'_F

Lemma (1 By direct computation)

The commutant of A_F in $M_8(\mathbb{C})$ is the algebra C_F with elements

$$\begin{bmatrix} q_{11} & & & q_{12} & \\ & \alpha & & & \\ & & \beta 1_2 & & \\ q_{21} & & & q_{22} & \\ & & & & \delta 1_3 \end{bmatrix}, \qquad (20)$$

where $\alpha, \beta, \delta \in \mathbb{C}$, $q = (q_{ij}) \in M_2(\mathbb{C})$.
The commutant of A_F in $\mathrm{End}_{\mathbb{C}}(H)$ is
$A'_F = C_F \otimes M_4(\mathbb{C}) \simeq M_4(\mathbb{C})^{\oplus 3} \oplus M_8(\mathbb{C})$, *of* $\dim_{\mathbb{C}} = 112$.

About the proof: $J_F A_F J_F$

$J_F A_F J_F \subset \mathrm{End}_{\mathbb{C}}(H_F)$ consists of elements of the form:

$$\begin{bmatrix} 1_4 & 0_4 \\ 0_4 & 0_4 \end{bmatrix} \triangleright \otimes \triangleleft \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array}\right] + \begin{bmatrix} 0_4 & 0_4 \\ 0_4 & 1_4 \end{bmatrix} \triangleright \otimes \triangleleft \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{q} \end{array}\right].$$

Note that A and $A_{\mathbb{C}}$ have the same commutant in $\mathrm{End}_{\mathbb{C}}(H_F)$. The map $a \mapsto J_F \bar{a} J_F$ gives two isomorphisms $A_F \to J_F A_F J_F$ and $(A_F)_{\mathbb{C}} \to (J_F A_F J_F)_{\mathbb{C}}$, and also the map $x \mapsto J_F \bar{x} J_F$ is an isomorphism between A'_F and $(J_F A_F J_F)'$.

About the proof: $(J_F A_F J_F)'$

From this,

Lemma

The commutant $(J_F A_F J_F)'$ of $J_F A_F J_F$ has elements

$$a \otimes e_{11} + \begin{bmatrix} b & \\ & c \end{bmatrix} \otimes e_{22} + \begin{bmatrix} b & \\ & d \end{bmatrix} \otimes (e_{33} + e_{44}) \tag{21}$$

with $a \in M_8(\mathbb{C})$, $b, c, d \in M_4(\mathbb{C})$.

Lemma

$A'_F \cap (J_F A_F J_F)' \simeq \mathbb{C}^{\oplus 10} \oplus M_2(\mathbb{C})$.

It follows that $\dim_{\mathbb{C}}(A'_F + (J_F A_F J_F)') = 210 \quad (= 2 \cdot 112 - 14)$.
The (real) subspace of hermitian matrices has $\dim_{\mathbb{R}} = 210$.

About the proof: $\mathcal{C}\ell_D(A)'$, Lemma A

Any unital complex $*$-subalgebra of $\mathrm{End}_{\mathbb{C}}(H)$, $\dim H < \infty$, is a finite direct sum of matrix algebras: $B \simeq \bigoplus_{i=1}^{s} M_{m_i}(\mathbb{C})$. Denote P_i the unit of $M_{m_i}(\mathbb{C})$, then $P_1, \ldots, P_s$ are orthogonal projections and H decomposes as $H \simeq \bigoplus_{i=1}^{s} H_i$, with

$$H_i = P_i \cdot H \simeq \mathbb{C}^{m_i} \otimes \mathbb{C}^{k_i} \ , \tag{22}$$

where k_i is multiplicity of the (unique) irrep $\mathbb{C}^{m_i}$ of $M_{m_i}(\mathbb{C})$ in H_i, and $M_{m_i}(\mathbb{C})$ acts on the 1st factor of $\mathbb{C}^{m_i} \otimes \mathbb{C}^{k_i}$ by matrix product.

Lemma (A)

The commutant of B in $\mathrm{End}_{\mathbb{C}}(H)$ is $B' \simeq \bigoplus_{i=1}^{s} M_{k_i}(\mathbb{C})$ and the action of B' on $H_i \simeq \mathbb{C}^{m_i} \otimes \mathbb{C}^{k_i}$ is given by matrix multiplication by $M_{k_i}(\mathbb{C})$ of the second factor in the tensor product.

About the proof: $\mathcal{C}\ell_D(A)'$, Lemma B

Lemma (B)

Let (A, H, D, J) *be a finite-dimensional real spectral triple and* $B \subseteq \mathrm{End}_{\mathbb{C}}(H)$ *a unital complex* $*$*-algebra satisfying:*

$$\mathcal{C}\ell_D(A) \subseteq B \qquad \text{and} \qquad B' = JBJ\ .$$

The following are equivalent:

(a) $\mathcal{C}\ell_D(A)' = J\mathcal{C}\ell_D(A)J$ *(the Hodge property)*

(b) $\mathcal{C}\ell_D(A)' \subseteq JBJ$

(c) $\mathcal{C}\ell_D(A) = B$.

Proof

(a)⇒(b) the hypothesis $\mathcal{C}\ell_D(A) \subseteq B$ *implies*

$J_F\mathcal{C}\ell_D(A)J_F \subseteq JBJ$;

and thus if from (a), i.e. $\mathcal{C}\ell_D(A)' = J_F\mathcal{C}\ell_D(A)J_F$, *follows (b)*

(b)⇒(c) $\mathcal{C}\ell_D(A)' \subseteq J_FBJ_F = B'$ *implies* $B \subseteq \mathcal{C}\ell_D(A)$ *and,*

since $\supseteq$ *holds by hypothesis,* $\mathcal{C}\ell_D(A) = B$.

(c)⇒(a) If (c), then $B' = J_FBJ_F$ *translates to*

About the proof: $\mathcal{C}\ell_D(A)'$

• Now, in our case we take

$$B := \mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \tag{23}$$

with $(\lambda, m, a, b) \in B$ represented on H_F as

$$\begin{bmatrix} \lambda & 0 \\ 0 & m \end{bmatrix} \otimes e_{22} \otimes 1 + a \otimes e_{11} \otimes e_{11} + b \otimes e_{11} \otimes (1 - e_{11}) \ , \tag{24}$$

• check that $\mathcal{C}\ell_{D_F}(A_F) \subset B$, i.e. 1st assumption of Lemma B
• check that B and JBJ commute, and so $JBJ \subseteq B'$.
• (24) is equivalent to the rep of B on (the 1st factors of):

$$(\mathbb{C} \otimes \mathbb{C}^4) \oplus (\mathbb{C}^3 \otimes \mathbb{C}^4) \oplus (\mathbb{C}^4 \otimes \mathbb{C}) \oplus (\mathbb{C}^4 \otimes \mathbb{C}^3)$$

given by matrix multiplication on the first factor.
• by Lemma A

$$B' \simeq M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \simeq B$$

and we have $JBJ = B'$, i.e. 2nd assumption of Lemma B
• find that $\mathcal{C}\ell_{D_F}(A_F)' \subseteq JBJ$, so get (b) and thus (a) of Lemma B which ends the proof.

Conclusions

The Connes-Chamseddine νSM interprets the geometry of the SM as gravity on the product of a (Riemannian) manifold M with a finite noncommutative 'internal' space F.
The multiplet of fundamental fermions (each a Dirac spinor on M) are fields on F that constitute H_F.

We show that the geometric nature of the latter is not a noncommutative analogue of Dirac spinors on F
(unless $>$2 new parameters are introduced in the matrix D_F,
so fields on M with physical status under scrutiny),
but rather of de-Rham forms on F (for one generation).

What happens for 3 generations (96×96 matrices) ?
Can be seen (not easily) that also then NO spin property,
and that $\mathcal{C}\ell_D(A)' \supset J\mathcal{C}\ell_D(A)J$ (order 2 condition).
Whether the Hodge property is satisfied is under investigation.

THANK YOU !

Refs

[CC12] A.H. Chamseddine and A. Connes, *Resilience of the Spectral Standard Model*, JHEP 1209 (2012) 104.
[C13] A. Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom. 7 (2013) 1–82.
[DD14] F. D'Andrea and L. Dąbrowski, *The Standard Model in Noncommutative Geometry and Morita equivalence*, J. Noncommut. Geom. 10 (2016) 551–578.
[DDS17] L. Dąbrowski, F. D'Andrea and A Sitarz, *The Standard Model in noncommutative geometry: fundamental fermions as internal forms*, arXiv:1703.05279
[Kra97] T. Krajewski, *Classification of Finite Spectral Triples*, J. Geom. Phys. 28 (1998) 1–30.
[LRV12] S. Lord, A. Rennie and J.C. Várilly, *Riemannian manifolds in noncommutative geometry*, J. Geom. Phys. 62 (2012) 1611–1638.
[PSS97] M. Paschke, F. Scheck and A. Sitarz, *Can (noncommutative) geometry accommodate leptoquarks?*, Phys. Rev. D59 (1999) 035003.