

Noncommutative products of Euclidean spaces

Giovanni Landi

Trieste

Mathematical models for noncommutativity in physics and quantum
spacetime

Bacach Center Warsaw, 2 Nov 2017.

Based on: M. Dubois-Violette, GL, arXiv:1706.06930

Abstract

- noncommutative generalizations of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$
- noncommutative generalizations of $\mathbb{S}^{N_1+N_2-1}$ and $\mathbb{S}^{N_1+N_2}$
- noncommutative generalizations of $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$
- a quaternionic noncommutative torus $\mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$, $\mathbf{u} \in \mathbb{S}^2 = \text{SU}(2)/\text{U}(1)$
- spherical manifolds : volume forms from top Chern–Connes characters
- noncommutative principal bundles
- spectral triples

The θ -deformation

\mathbb{C}_θ^2 = Noncommutative space dual to the
*-algebra \mathcal{A}_θ generated by normal elements z_1, z_2 with relations

$$z_1 z_2 = e^{i\theta} z_2 z_1, \quad z_1 z_2^* = e^{-i\theta} z_2^* z_1 \quad (1)$$

\Rightarrow center generated by $z_1 z_1^* = \|z_1\|^2$ and $z_2 z_2^* = \|z_2\|^2$

Real version $\mathbb{C}_\theta^2 = (\mathbb{R}^2)_\theta^2 = \mathbb{R}^2 \times_\theta \mathbb{R}^2$

$$z_1 = x_1^1 + i x_1^2, \quad z_2 = x_2^1 + i x_2^2, \quad (x_k^\lambda)^* = x_k^\lambda$$

and $\{(1) + \text{normality of } z_k\} \Leftrightarrow (2)$

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda & ; & x_2^\lambda x_2^\mu = x_2^\mu x_2^\lambda \\ x_1^\lambda x_2^\mu = R_{\nu\rho}^{\lambda\mu} x_2^\nu x_1^\rho \end{cases} \quad (2)$$

with

$$R_{\nu\rho}^{\lambda\mu} = \cos(\theta) \delta_\rho^\lambda \delta_\nu^\mu + i \sin(\theta) C_\rho^\lambda D_\nu^\mu \quad C = -D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

2-torus and 3-sphere

$$\mathbb{R}^2 \times_{\theta} \mathbb{R}^2 / (\|x_1\|^2 - 1, \|x_2\|^2 - 1) = \mathbb{T}_{\theta}^2$$

$$\mathbb{R}^2 \times_{\theta} \mathbb{R}^2 / (\|x_1\|^2 + \|x_2\|^2 - 1) = \mathbb{S}_{\theta}^3$$

nice examples of singular spaces

The quadratic $*$ -algebra \mathcal{A}_R

The $*$ -algebra \mathcal{A}_R is generated by two sets of hermitian elements x_1^λ with $\lambda \in \{1, \dots, N_1\}$ and x_2^α with $\alpha \in \{1, \dots, N_2\}$ with relations

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda & ; & x_2^\alpha x_2^\beta = x_2^\beta x_2^\alpha \\ x_1^\lambda x_2^\alpha = R_{\beta\mu}^{\lambda\alpha} x_2^\beta x_1^\mu \end{cases} \quad (3)$$

for suitable $R_{\beta\mu}^{\lambda\alpha} \in \mathbb{C}$. In view of the hermiticity of the x_1^λ, x_2^α we impose

$$\overline{R_{\beta\mu}^{\lambda\alpha}} R_{\gamma\nu}^{\mu\beta} = \delta_\nu^\lambda \delta_\gamma^\alpha \quad (4)$$

Thus \mathcal{A}_R is a graded quadratic algebra $\mathcal{A}_R = \bigoplus_n \mathcal{A}_R^n$

which is connected: $\mathcal{A}_R^0 = \mathbb{C}\mathbf{1}$;

the elements x_1^λ, x_2^α form a basis of \mathcal{A}_R^1 ;

by the requirement (4):

the elements $x_1^\lambda x_1^\mu$ with $\lambda \leq \mu$, $x_2^\alpha x_2^\beta$ with $\alpha \leq \beta$ and $x_1^\lambda x_2^\alpha$ form a basis of \mathcal{A}_R^2 .

The Koszul dual $\mathcal{A}_R^!$

A quadratic algebra is an algebra \mathcal{A} of the form

$$\mathcal{A} = T(E)/(\mathcal{R})$$

where E is a finite-dimensional vector space and (\mathcal{R}) denotes the ideal of the tensor algebra generated by $\mathcal{R} \subset E \otimes E$.

The Koszul dual $\mathcal{A}^!$ of \mathcal{A} is the quadratic algebra

$$\mathcal{A}^! = T(E^*)/(\mathcal{R}^\perp)$$

where $\mathcal{R}^\perp \subset E^* \otimes E^*$ is the orthogonal of \mathcal{R} .

Our $\mathcal{A}_R^!$ is generated by the dual bases $\theta_\lambda^1, \theta_\alpha^2$ of the x_1^λ, x_2^α with relations

$$\begin{cases} \theta_\lambda^1 \theta_\mu^2 = -\theta_\mu^2 \theta_\lambda^1 & ; & \theta_\alpha^2 \theta_\beta^2 = -\theta_\beta^2 \theta_\alpha^2 \\ \theta_\beta^2 \theta_\mu^1 = -R_{\beta\mu}^{\lambda\alpha} \theta_\lambda^1 \theta_\alpha^2 \end{cases} \quad (5)$$

More on Koszul algebras

A (homogeneous) *quadratic algebra* is an associative algebra \mathcal{A} of the form

$$\mathcal{A} = A(E, R) = T(E)/(R)$$

Here E is a finite-dimensional vector space (the space of generators), R a subspace of $E \otimes E$ (the space of relations of \mathcal{A})

The algebra $\mathcal{A} = A(E, R)$ is a connected (i.e. $\mathcal{A}_0 = \mathbb{C}\mathbf{1}$) graded algebra $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$ generated by the degree 1 part, $\mathcal{A}_1 = E$.

Its *Koszul dual* $\mathcal{A}^!$, is the quadratic algebra defined by

$$\mathcal{A}^! = A(E^*, R^\perp)$$

where E^* denotes the dual vector space of E and $R^\perp \subset E^* \otimes E^*$ is the orthogonal of the space of relations $R \subset E \otimes E$: $R^\perp = \{\omega \in E^* \otimes E^* ; \langle \omega, r \rangle = 0\}$.

The dual vector spaces $\mathcal{A}_n^{!*}$ of the homogeneous components $\mathcal{A}_n^!$ of $\mathcal{A}^!$ are

$$\mathcal{A}_1^{!*} = E \quad \text{and} \quad \mathcal{A}_n^{!*} = \bigcap_{r+s+2=n} E^{\otimes r} \otimes R \otimes E^{\otimes s}$$

In particular $\mathcal{A}_2^{!*} = R$ and $\mathcal{A}_n^{!*} \subset E^{\otimes n}$ for any $n \in \mathbb{N}$.

The sequence of free left \mathcal{A} -modules

$$\cdots \xrightarrow{b} \mathcal{A} \otimes \mathcal{A}_{n+1}^{!*} \xrightarrow{b} \mathcal{A} \otimes \mathcal{A}_n^{!*} \rightarrow \cdots \rightarrow \mathcal{A} \otimes \mathcal{A}_2^{!*} \xrightarrow{b} \mathcal{A} \otimes E \xrightarrow{b} \mathcal{A} \rightarrow 0 \quad (6)$$

where $b : \mathcal{A} \otimes \mathcal{A}_{n+1}^{!*} \rightarrow \mathcal{A} \otimes \mathcal{A}_n^{!*}$ defined by

$$b(a \otimes (x_0 \otimes x_1 \otimes \cdots \otimes x_n)) = ax_0 \otimes (x_1 \otimes \cdots \otimes x_n)$$

is such that $b^2 = 0$, is a chain complex $K(\mathcal{A})$ of free left \mathcal{A} -modules called the *Koszul complex* of the quadratic algebra \mathcal{A} .

The quadratic algebra \mathcal{A} is said to be a *Koszul algebra* whenever its Koszul complex is acyclic in positive degrees, that is, $H_n(K(\mathcal{A})) = 0$ for $n \geq 1$.

The presentation of \mathcal{A} by generators and relations is equivalent to the exactness of the sequence

$$\mathcal{A} \otimes R \xrightarrow{b} \mathcal{A} \otimes E \xrightarrow{b} \mathcal{A} \xrightarrow{\varepsilon} \mathbb{C} \rightarrow 0 \quad (7)$$

so one always has $H_1(K(\mathcal{A})) = 0$ and $H_0(K(\mathcal{A})) = \mathbb{C}$ and, whenever \mathcal{A} is Koszul, the sequence

$$K(\mathcal{A}) \xrightarrow{\varepsilon} \mathbb{C} \rightarrow 0, \quad (8)$$

with ε induced by the projection onto degree 0, is a free resolution of the trivial module \mathbb{C} ; a minimal projective resolution of \mathbb{C} of graded modules.

If $\mathcal{A} = A(E, R)$ quadratic Koszul is such that $\mathcal{A}_D^! \neq 0$ and $\mathcal{A}_n^! = 0$ for $n > D$, then the trivial (left) module \mathbb{C} has projective dimension D which implies that \mathcal{A} has global dimension D . In turn the Hochschild dimension of \mathcal{A} is D .

By applying the functor $Hom_{\mathcal{A}}(\cdot, \mathcal{A})$ to the Koszul chain complex $K(\mathcal{A})$ of left \mathcal{A} -modules one obtains the cochain complex $L(\mathcal{A})$ of right \mathcal{A} -modules

$$0 \rightarrow \mathcal{A} \xrightarrow{b'} \cdots \xrightarrow{b'} \mathcal{A}_n^! \otimes \mathcal{A} \xrightarrow{b'} \mathcal{A}_{n+1}^! \otimes \mathcal{A} \xrightarrow{b'} \cdots . \quad (9)$$

with b' left multiplication by $\sum_k \theta^k \otimes e_k$ in $\mathcal{A}^! \otimes \mathcal{A}$ and (e_k, θ^k) are dual bases.

The algebra \mathcal{A} is said to be *Koszul-Gorenstein* if it is Koszul of finite global dimension D as above and if $H^n(L(\mathcal{A})) = \mathbb{C} \delta_D^n$.

This implies that $\mathcal{A}_n^! \simeq \mathcal{A}_{D-n}^{!*}$ as vector spaces (a version of Poincaré duality).

Finally, a graded algebra $\mathcal{A} = \bigoplus_n \mathcal{A}_n$ is said to have *polynomial growth* whenever there are a positive C and $N \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$,

$$\dim(\mathcal{A}_n) \leq Cn^{N-1}.$$

An \mathcal{R} -matrix

Define x^a for $a \in \{1, \dots, N_1 + N_2\}$ by $x^\lambda = x_1^\lambda$, $x^{\alpha+N_1} = x_2^\alpha$.

Then the relations (3) together with $x_2^\alpha x_1^\lambda = \overline{R}_{\beta\mu}^{\lambda\alpha} x_1^\mu x_2^\alpha$ reads

$$x^a x^b = \mathcal{R}_{cd}^{ab} x^c x^d \quad (10)$$

In view of (3) and (4) one gets:

$$\begin{aligned} \mathcal{R}_{\tau\rho}^{\lambda\mu} &= \delta_\rho^\lambda \delta_\tau^\mu, & \mathcal{R}_{\alpha\beta}^{\gamma\delta} &= \delta_\beta^\gamma \delta_\alpha^\delta \\ \mathcal{R}_{\beta\mu}^{\lambda\alpha} &= R_{\beta\mu}^{\lambda\alpha}, & \mathcal{R}_{\mu\beta}^{\alpha\lambda} &= \overline{R}_{\beta\mu}^{\lambda\alpha} \end{aligned}$$

$$\begin{aligned} \mathcal{R}_{\alpha\nu}^{\lambda\mu} &= \mathcal{R}_{\alpha\beta}^{\lambda\mu} = \mathcal{R}_{\nu\beta}^{\lambda\mu} = 0, \\ \mathcal{R}_{\lambda\beta}^{\alpha\gamma} &= \mathcal{R}_{\lambda\mu}^{\alpha\gamma} = \mathcal{R}_{\beta\mu}^{\alpha\gamma} = 0, \\ \mathcal{R}_{\mu\nu}^{\lambda\alpha} &= \mathcal{R}_{\beta\gamma}^{\lambda\alpha} = \mathcal{R}_{\mu\beta}^{\lambda\alpha} = 0, \\ \mathcal{R}_{\mu\nu}^{\alpha\lambda} &= \mathcal{R}_{\beta\gamma}^{\alpha\lambda} = \mathcal{R}_{\beta\mu}^{\alpha\lambda} = 0. \end{aligned}$$

Yang-Baxter condition

The matrix \mathcal{R} is involutive, i.e. $\mathcal{R}^2 = \mathbf{1} \otimes \mathbf{1}$ or

$$\mathcal{R}_{cd}^{ab} \mathcal{R}_{ef}^{cd} = \delta_e^a \delta_f^b$$

We next impose the Yang-Baxter condition for \mathcal{R}

$$(\mathcal{R} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbf{1})^{abc} = (\mathbf{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{R})^{abc}$$

Yang-Baxter condition - cont.d

This breaks in a series of conditions:

The cubic YB equations are equivalent to quadratic relations:

$$\left\{ \begin{array}{l} R_{\gamma\rho}^{\lambda\alpha} R_{\delta\mu}^{\rho\beta} = R_{\delta\rho}^{\lambda\beta} R_{\gamma\mu}^{\rho\alpha} \quad (\lambda\alpha\beta) \\ \bar{R}_{\gamma\rho}^{\lambda\alpha} \bar{R}_{\delta\mu}^{\rho\beta} = \bar{R}_{\delta\rho}^{\lambda\beta} \bar{R}_{\gamma\mu}^{\rho\alpha} \quad (\alpha\beta\lambda) \\ \bar{R}_{\gamma\rho}^{\lambda\alpha} R_{\delta\mu}^{\rho\beta} = R_{\delta\rho}^{\lambda\beta} \bar{R}_{\gamma\mu}^{\rho\alpha} \quad (\alpha\lambda\beta) \end{array} \right. \quad (11)$$

and

$$\left\{ \begin{array}{l} R_{\gamma\nu}^{\lambda\alpha} R_{\beta\rho}^{\mu\gamma} = R_{\gamma\rho}^{\mu\alpha} R_{\beta\nu}^{\lambda\gamma} \quad (\lambda\mu\alpha) \\ \bar{R}_{\gamma\nu}^{\lambda\alpha} \bar{R}_{\beta\rho}^{\mu\gamma} = \bar{R}_{\gamma\rho}^{\mu\alpha} \bar{R}_{\beta\nu}^{\lambda\gamma} \quad (\alpha\lambda\mu) \\ R_{\gamma\nu}^{\lambda\alpha} \bar{R}_{\beta\rho}^{\mu\gamma} = \bar{R}_{\gamma\rho}^{\mu\alpha} R_{\beta\nu}^{\lambda\gamma} \quad (\lambda\alpha\mu) \end{array} \right. \quad (12)$$

for the $R_{\beta\mu}^{\lambda\alpha}$ and $\bar{R}_{\beta\mu}^{\lambda\alpha}$ (the components abc of (7) are in the (...)).

Noncommutative product of \mathbb{R}^{N_1} and \mathbb{R}^{N_2}

From now on it is assumed that the matrix R of relations (3) for the algebra \mathcal{A}_R satisfies conditions (4), (11) and (12)

The classical (commutative) solution $R = R_0$ is

$$(R_0)_{\beta\mu}^{\lambda\alpha} = \delta_\mu^\lambda \delta_\beta^\alpha$$

and the corresponding algebra \mathcal{A}_{R_0} is the algebra of polynomial functions on the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

This is the reason we define the **noncommutative product of \mathbb{R}^{N_1} and \mathbb{R}^{N_2}**

$$\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}$$

to be the “dual” of the algebra \mathcal{A}_R for general R .

Regularity properties of \mathcal{A}_R and $\mathcal{A}_R^!$

Since the relations of \mathcal{A}_R can be written in the form (6) with \mathcal{R} involutive and satisfying the Yang-Baxter equation,

it follows from general results (Gurevich, Wambst) that \mathcal{A}_R is very regular.

In particular \mathcal{A}_R is a Koszul algebra of global dimension $N_1 + N_2$ having the Gorenstein property (an appropriate version of the Poincaré duality property)

In our case, this implies that, in terms of the x^a and the dual basis θ_a :

the $x^{a_1} \dots x^{a_p}$ for $a_1 \leq \dots \leq a_p$ and $p \in \mathbb{N}$ is a basis of \mathcal{A}_R

while the $\theta_{a_1} \dots \theta_{a_p}$ for $a_1 < \dots < a_p$ and $p \in \{1, \dots, N_1 + N_2\}$ is a basis of $\mathcal{A}_R^!$.

As a consequence the Poincaré series are classical :

$$P_{\mathcal{A}_R}(t) := \sum_n \dim(\mathcal{A}_R^n) t^n = \left(\frac{1}{1-t} \right)^{N_1+N_2} \quad \text{and} \quad P_{\mathcal{A}_R^!}(t) = (1+t)^{N_1+N_2} \quad (13)$$

Regularity properties cont.d

The algebra \mathcal{A}_R is even a Calabi–Yau algebra:

any generator of the top one-dimensional space $((\mathcal{A}_R^!)_{N_1+N_2})^*$

is a cyclic potential for the algebra \mathcal{A}_R (Ginzburg)

(a cyclic pre-regular multilinear form)

Noncommutative product of Euclidean spaces

Theorem. The following conditions (i) (ii) and (iii) are equivalent :

$$(i) \sum_{a=1}^{N_1+N_2} (x^a)^2 = \sum_{\lambda=1}^{N_1} (x_1^\lambda)^2 + \sum_{\alpha=1}^{N_2} (x_2^\alpha)^2 \quad \text{is central in } \mathcal{A}_R,$$

$$(ii) \sum_{\lambda=1}^{N_1} (x_1^\lambda)^2 \quad \text{and} \quad \sum_{\alpha=1}^{N_2} (x_2^\alpha)^2 \quad \text{are in the center of } \mathcal{A}_R, \quad (14)$$

$$(iii) \sum_{\lambda=1}^{N_1} R_{\beta\nu}^{\lambda\gamma} R_{\alpha\mu}^{\lambda\beta} = \delta_\alpha^\gamma \delta_{\mu\nu} \quad \text{and} \quad \sum_{\alpha=1}^{N_2} R_{\beta\rho}^{\lambda\alpha} R_{\gamma\mu}^{\rho\alpha} = \delta_\mu^\lambda \delta_{\beta\gamma}$$

We take R to satisfy also (14) and define the **the noncommutative product of the Euclidean space \mathbb{R}^{N_1} with the Euclidean space \mathbb{R}^{N_2}** to be dual of \mathcal{A}_R .

Clearly, the relations (11) are satisfied by the classical $R = R_0$;

\mathcal{A}_R generalizes the algebra of polynomial functions on the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

Restrictions on the structure of R

By using (4) and (14) one obtains

$$R_{\alpha\mu}^{\lambda\beta} = R_{\beta\lambda}^{\mu\alpha} = \overline{R}_{\alpha\lambda}^{\mu\beta} = (R^{-1})_{\lambda\alpha}^{\beta\mu} \quad (15)$$

In turn this implies that relations (11) and (12) reduce to

$$R_{\beta\rho}^{\lambda\alpha} R_{\gamma\mu}^{\rho\delta} = R_{\gamma\rho}^{\lambda\delta} R_{\beta\mu}^{\rho\alpha} \quad \text{and} \quad R_{\gamma\nu}^{\lambda\alpha} R_{\beta\rho}^{\mu\gamma} = R_{\gamma\rho}^{\mu\alpha} R_{\beta\nu}^{\lambda\gamma} \quad (16)$$

that is the first relation of (11) and the first relation of (12).

Corollary. Relations (3) define (the algebra of) a **noncommutative product** of a N_1 -dimensional with a N_2 -dimensional Euclidean spaces if and only if the $R_{\beta\mu}^{\lambda\alpha}$ satisfy relations (15) and (16).

Noncommutative product of spheres

The elements $\sum_{\lambda=1}^{N_1} (x_1^\lambda)^2 = \|x_1\|^2$ and $\sum_{\alpha=1}^{N_2} (x_2^\alpha)^2 = \|x_2\|^2$ of \mathcal{A}_R being central one may consider the quotient algebra

$$\mathcal{A}_R / (\{\|x_1\|^2 - \mathbf{1}, \|x_2\|^2 - \mathbf{1}\}) \quad \leftrightarrow \quad \mathbb{S}^{N_1-1} \times_R \mathbb{S}^{N_2-1}$$

This defines by duality the noncommutative product of the classical spheres \mathbb{S}^{N_1-1} and \mathbb{S}^{N_2-1} .

Indeed, for $R = R_0$, the above quotient is the restriction to $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$ of the polynomial functions on $\mathbb{R}^{N_1+N_2}$.

Noncommutative spheres

With $\|x\|^2$ denoting the central element $\sum_{a=1}^{N_1+N_2} (x^a)^2 = \|x_1\|^2 + \|x_2\|^2$, one may also consider the quotient of \mathcal{A}_R

$$\mathcal{A}_R / (\|x\|^2 - \mathbf{1}) \quad \leftrightarrow \quad (\mathbb{S}^{N_1+N_2-1})_R$$

This defines (dualy) the noncommutative $(N_1 + N_2 - 1)$ -sphere $(\mathbb{S}^{N_1+N_2-1})_R$ (a subspace of the noncommutative product of \mathbb{R}^{N_1} with \mathbb{R}^{N_2})

This is a **noncommutative spherical manifold** in the sense of Connes–Landi and Connes–Dubois-Violette (see below).

The (generalized) Clifford algebra $Cl(\mathcal{A}_R)$

The $*$ -algebra $Cl(\mathcal{A}_R)$ is generated by two sets of hermitian elements Γ_λ^1 with $\lambda \in \{1, \dots, N_1\}$ and Γ_α^2 with $\alpha \in \{1, \dots, N_2\}$ with relations

$$\begin{cases} \Gamma_\lambda^1 \Gamma_\mu^1 + \Gamma_\mu^1 \Gamma_\lambda^1 = 2\delta_{\lambda\mu} \mathbf{1} \\ \Gamma_\alpha^2 \Gamma_\beta^2 + \Gamma_\beta^2 \Gamma_\alpha^2 = 2\delta_{\alpha\beta} \mathbf{1} \\ \Gamma_\beta^2 \Gamma_\mu^1 + R_{\beta\mu}^{\lambda\alpha} \Gamma_\lambda^1 \Gamma_\alpha^2 = 0 \end{cases} \quad (17)$$

Proposition In the algebra $Cl(\mathcal{A}_R) \otimes \mathcal{A}_R$ one has :

$$(\Gamma_\lambda^1 \otimes x_1^\lambda)^2 = \mathbf{1} \otimes \|x_1\|^2, \quad (\Gamma_\alpha^2 \otimes x_2^\alpha)^2 = \mathbf{1} \otimes \|x_2\|^2$$

and

$$(\Gamma_\lambda^1 \otimes x_1^\lambda)(\Gamma_\alpha^2 \otimes x_2^\alpha) + (\Gamma_\alpha^2 \otimes x_2^\alpha)(\Gamma_\lambda^1 \otimes x_1^\lambda) = 0$$

Structure of $Cl(\mathcal{A}_R)$

Last proposition is equivalent to

$$(\Gamma(x))^2 = \mathbf{1} \otimes \|x\|^2 \quad (18)$$

with $\Gamma(x) = \Gamma_a \otimes x^a = \Gamma_\lambda^1 \otimes x_1^\lambda + \Gamma_\alpha^2 \otimes x_2^\alpha$ and $\|x\|^2 = \sum_{a=1}^{N_1+N_2} (x^a)^2$.

The algebra $Cl(\mathcal{A}_R)$ is nonhomogeneous quadratic with $\mathcal{A}_R^!$ as homogeneous part. It is not \mathbb{N} -graded but only \mathbb{Z}_2 -graded and filtered with

$$\mathcal{F}^n = F^n(Cl(\mathcal{A}_R)) = \{\text{elements of degree in } \Gamma \leq n\}$$

One has a surjective canonical homomorphism of graded algebra

$$\text{can} : \mathcal{A}_R^! \rightarrow \text{gr}(Cl(\mathcal{A}_R^!)) = \bigoplus_{n \in \mathbb{N}} \mathcal{F}^n / \mathcal{F}^{n-1} \quad (19)$$

which induce the isomorphism of vector spaces

$$(\mathcal{A}_R^!)^1 \simeq \mathcal{F}^1 / \mathcal{F}^0$$

Structure of $Cl(\mathcal{A}_R)$ - cont.d

The fact that $\|x\|^2 = \sum (x^a)^2$ is central in \mathcal{A}_R and the Koszulity of $\mathcal{A}_R^!$ imply the following PBW property (via the duality of Positselski).

Proposition The homomorphism (19) is an isomorphism of graded algebras.

Thus $Cl(\mathcal{A}_R)$ is a Koszul nonhomogeneous quadratic algebra since $\mathcal{A}_R^!$ is Koszul, (cf. Dubois-Violette). This implies

$$\dim Cl(\mathcal{A}_R) = \dim \mathcal{A}_R^! = 2^{N_1+N_2}$$

One has the following isomorphisms :

$$\left\{ \begin{array}{l} Cl(N_1) \simeq \text{subalgebra of } Cl(\mathcal{A}) \text{ generated by the } \Gamma_\lambda^1 \\ Cl(N_2) \simeq \text{subalgebra of } Cl(\mathcal{A}) \text{ generated by the } \Gamma_\alpha^2 \end{array} \right.$$

and

$$Cl(\mathcal{A}_R) \simeq Cl(N_1 + N_2) \tag{20}$$

The general solution

Let us introduce a two-index $\binom{\lambda}{\alpha}$ with $\lambda \in \{1, \dots, N_1\}$ and $\alpha \in \{1, \dots, N_2\}$. Then the conditions (15) can be read as symmetry conditions:

$$R_{(\beta)}^{(\lambda)} = R_{(\alpha)}^{(\mu)} = \bar{R}_{(\lambda)}^{(\mu)} = (R^{-1})_{(\beta)}^{(\mu)}.$$

While, the quadratic conditions in (16) can be written as

$$R_{(\beta)}^{(\lambda)} R_{(\delta)}^{(\rho)} = R_{(\delta)}^{(\lambda)} R_{(\beta)}^{(\rho)} \quad \text{and} \quad R_{(\gamma)}^{(\lambda)} R_{(\rho)}^{(\mu)} = R_{(\rho)}^{(\lambda)} R_{(\gamma)}^{(\mu)}.$$

Theorem The general solution is of the form

$$R_{\alpha\mu}^{\lambda\beta} = S_{\mu\alpha}^{\lambda\beta} + i A_{\mu\alpha}^{\lambda\beta}$$

$$\text{with} \quad [S, A] = 0 \quad S^2 + A^2 = \mathbf{1}_{N_1} \otimes \mathbf{1}_{N_2}.$$

S and A are real matrices with S standing for symmetric while A antisymmetric for exchange in indices $\lambda \leftrightarrow \mu$ and $\alpha \leftrightarrow \beta$

The ansatz $A B C D$

Nontrivial realizations of the $R_{\beta\mu}^{\lambda\alpha}$ are given by the following.

Proposition Let A and C be two commuting real $N_1 \times N_1$ -matrices with A symmetric and C antisymmetric and let B and D be two commuting real $N_2 \times N_2$ -matrices with B symmetric and D antisymmetric. Assume that

$$A^2 \otimes B^2 + C^2 \otimes D^2 = \mathbf{1}_{N_1} \otimes \mathbf{1}_{N_2}$$

then the $R_{\beta\mu}^{\lambda\alpha}$ given by

$$R_{\beta\mu}^{\lambda\alpha} = A_{\mu}^{\lambda} B_{\beta}^{\alpha} + i C_{\mu}^{\lambda} D_{\beta}^{\alpha} \quad (21)$$

satisfy the assumptions (15), (16).

Enter the quaternions

An $SO(4)$ -invariant decomposition of $M_4(\mathbb{R})$

$$q = x^0 \mathbf{1} + x^k e_k \in \mathbb{H} \quad \longleftrightarrow \quad x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \quad \text{Euclidean}$$

a right and a left action

$$e_k q \leftrightarrow J_k^{(+)} x, \quad q e_k \leftrightarrow -J_k^{(-)} x$$

$$J_{k\mu\nu}^{(\pm)} = \mp (\delta_{0\mu} \delta_{k\nu} - \delta_{0\nu} \delta_{k\mu}) - \varepsilon_{klm} \delta_\mu^\ell \delta_\nu^m$$

$$J_k^{(\pm)} J_\ell^{(\pm)} = -\delta_{kl} \mathbf{1} + \sum_m \varepsilon_{klm} J_m^{(\pm)}, \quad J_k^{(+)} J_\ell^{(-)} = J_\ell^{(-)} J_k^{(+)}$$

$$M_4(\mathbb{R}) = \mathbb{R}^4 \otimes \mathbb{R}^4 = \mathbb{R} \mathbf{1} \oplus \wedge_{(+)}^2 \mathbb{R}^4 \oplus \wedge_{(-)}^2 \mathbb{R}^4 \oplus \mathbb{S}_0^2 \mathbb{R}^4$$

Orthornormal basis $\mathbf{1}, J_k^{(+)}, J_\ell^{(-)}, J_r^{(+)} J_s^{(-)}$

$(+)$ = antisymmetric self-dual, $(-)$ = antisymmetric anti-self dual.

Noncommutative quaternionic planes

Use last theorem for $N_1 = N_2 = 4$

$$A = \mathbf{1}, \quad B = u^0 \mathbf{1}, \quad C = J_1^{(\pm)}, \quad D = u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)}$$

with $(u^0)^2 + (u^1)^2 + (u^2)^2 = 1$. This gives:

$$R_{\beta\mu}^{\lambda\alpha} = u^0 \delta_\mu^\lambda \delta_\beta^\alpha + i (J_1^{(\pm)})_\mu^\lambda (u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)})_\beta^\alpha \quad (22)$$

By using the $J_k^{(\mp)}$ one defines an action of \mathbb{H} .

The choice of the direction 1 and of the plane (1 2) is immaterial since one can change them into an arbitrary direction \vec{n} and an arbitrary plane which contains \vec{n} by a rotation of $SO_3^{(\pm)}$.

The exchange $(+) \leftrightarrow (-)$ is induced for instance by the exchange $x^0 \leftrightarrow -x^0$ and therefore does not change the algebra \mathcal{A}_R for R given by (22).

Noncommutative quaternionic planes cont.d

The solution given by (22) generalizes \mathbb{C}_θ^2 for $\mathbb{C} \rightarrow \mathbb{H}$;

For the θ -deformation the parameter is in fact

$$\mathbb{S}^1/\mathbb{S}^0 = U_1(\mathbb{C})/U_1(\mathbb{R}) = P_1(\mathbb{R}).$$

The parameter here

$$\mathbf{u} \in \mathbb{S}^2 = \mathbb{S}^3/\mathbb{S}^1 = U_1(\mathbb{H})/U_1(\mathbb{C}) = P_1(\mathbb{C})$$

and for $u^0 = 1$ ($\Rightarrow u^1 = u^2 = 0$), this gives the classical \mathbb{H}^2 .

Noncommutative quaternionic tori

The N.C. product of \mathbb{H} by \mathbb{H} corresponding to \mathcal{A}_R is denoted $\mathbb{H}_{\mathbf{u}}^2$.

Tori obtained by the quotient by the ideal generated by $\{\|x_1\|^2 - 1, \|x_2\|^2 - 1\}$:

$$\mathcal{A}(\mathbb{T}_{\mathbf{u}}^{\mathbb{H}}) = \mathcal{A}(\mathbb{H}_{\mathbf{u}}^2) / \langle \|x_1\|^2 - 1, \|x_2\|^2 - 1 \rangle$$

$$\mathbb{T}_{\mathbf{u}}^{\mathbb{H}} \simeq \mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$$

an $SU(2) \times SU(2)$ action

Additional strata: other N.C. products of 4-dim. Euclidean spaces

Other \mathcal{A}_R with $N_1 = N_2 = 4$ using the ansatz $A B C D$ with $J_k^{(\pm)}$

$$1. A = \mathbf{1}, \quad B = \cos(\theta)\mathbf{1}, \quad C = J_1^{(\pm)}, \quad D = \sin(\theta)J_1^{(\mp)}.$$

The direction $\mathbf{1}$ for $J_k^{(\pm)}$ (resp $J_k^{(\mp)}$) can be changed by acting with $SO_3^{(\pm)}$ (resp. $SO_3^{(\mp)}$). For $\theta = 0$ it corresponds to the classical $\mathbb{R}^4 \times \mathbb{R}^4$.

$$2. A = C \cdot (v^k J_k^{(\mp)}), \quad B = D \cdot (w^k J_k^{(\mp)}), \\ C = J_1^\pm, \quad D = u^1 J_1^\pm + u^2 J_2^\pm \quad \text{with } ((u^1)^2 + (u^2)^2)(1 + \vec{v}^2 \vec{w}^2) = 1.$$

$$3. A = C \cdot (v^k J_k^{(\mp)}), \quad B = D \cdot (w^k J_k^\pm), \\ C = J_1^{(\pm)}, \quad D = u J_1^{(\mp)} \quad \text{with } u^2 + \vec{v}^2 \vec{w}^2 = 1.$$

Solutions 2 and 3 do not contain the classical $\mathbb{R}^4 \times \mathbb{R}^4$.

These solutions are noncommutative for any parameters and cannot be connected with solution 1 and with \mathbb{H}_u^2 .

Spherical conditions

A projection $p \in M_{2^n}(\mathcal{A}(S_R^{2n}))$,

$$p = \frac{1}{2} (\mathbf{1} + \Gamma_a x^a + \Gamma x)$$

$$ch_k(p) = 0, \quad 0 \leq k \leq n - 1 \quad ch_n(p) \quad \text{the volume form}$$

A unitary $U \in M_{2^{n-1}}(\mathcal{A}(S_R^{2n-1}))$,

$$U = \mathbf{1} x^0 + \Sigma_j x^j$$

$$ch_{k-\frac{1}{2}}(U) = 0, \quad 0 \leq k \leq n - 1 \quad ch_{n-\frac{1}{2}}(U) \quad \text{the volume form}$$

$$\Gamma_\mu = \begin{pmatrix} 0 & \Sigma_\mu \\ \Sigma_\mu, & 0 \end{pmatrix}$$

A matter of computation for the present examples

Principal bundles

Consider the two quaternions.

$$x_1 = x_1^\mu e_\mu \quad x_2 = x_2^\alpha e_\alpha$$

with commutation relations governed by a matrix $R_{\beta\mu}^{\lambda\alpha}$. When restricting to the sphere \mathbb{S}_R^7 , we get a normalised vector valued function

$$|\psi\rangle = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \quad \langle\psi, \psi\rangle = \|x_1\|^2 + \|x_2\|^2 = \mathbf{1}$$

and thus a projection: ($p = p^* = p^2$)

$$p = |\psi\rangle \langle\psi| = \begin{pmatrix} x_2 x_2^* & x_2 x_1^* \\ x_1 x_2^* & x_1 x_1^* \end{pmatrix},$$

Define coordinate functions $Y = Y^0 e_0 + Y^k e_k$ and Y^4 by

$$Y^4 = \|x_2\|^2 - \|x_1\|^2 \quad \text{and} \quad \frac{1}{2}Y = x_2 x_1^*$$

so that

$$p = |\psi\rangle \langle\psi| = \frac{1}{2} \begin{pmatrix} 1 + Y^4 & Y \\ Y^* & 1 - Y^4 \end{pmatrix}.$$

The condition $p^2 = p$ leads to

$$YY^* + (Y^4)^2 = 1 \quad \text{and} \quad Y^*Y + (Y^4)^2 = 1$$

$$YY^4 = Y^4Y \quad \text{and} \quad Y^*Y^4 = Y^4Y^*.$$

Thus the coordinate function Y^4 is central while comparing the first two conditions requires $YY^* = Y^*Y$ and that this is a multiple of the identity. These translates to the following conditions

$$\begin{aligned} \sum_{\mu=0}^3 (Y^{\mu*}Y^{\mu} - Y^{\mu}Y^{\mu*}) &= 0, \\ -(Y^{0*}Y^k - Y^{k*}Y^0) + \varepsilon_{kmn}Y^{m*}Y^n &= 0, \\ Y^0Y^{k*} - Y^kY^{0*} + \varepsilon_{kmn}Y^mY^{n*} &= 0 \end{aligned}$$

for $k, r, m = 1, 2, 3$ and totally antisymmetric tensor ε_{krm} .

Then the first condition at top of the page reduces to a four-sphere relation

$$\sum_{\mu=0}^3 Y^{\mu*}Y^{\mu} + (Y^4)^2 = 1 = \sum_{\mu=0}^3 Y^{\mu*}Y^{\mu} + (Y^4)^2 = 1. \quad (23)$$

The elements Y^μ generate the $*$ -algebra $\mathcal{A}(\mathbb{S}_R^4)$ of a 4-sphere \mathbb{S}_R^4 .

The algebra inclusion $\mathcal{A}(\mathbb{S}_R^4) \hookrightarrow \mathcal{A}(\mathbb{S}_R^7)$ is a principal $SU(2)$ bundle:

A unit quaternion $w \in \mathbb{H}_1 \simeq SU(2)$ act on \mathbb{S}_R^7 as

$$\alpha_w(|\psi\rangle) = |\psi\rangle w = \begin{pmatrix} x_2 w \\ x_1 w \end{pmatrix}.$$

leaving the projection p and then the algebra $\mathcal{A}(\mathbb{S}_R^4)$ invariant.

If $H = \mathcal{A}(SU(2))$ we have dually a co-action δ of H on $\mathcal{A}(\mathbb{S}_R^7)$ with algebra of co-invariant element again the subalgebra $\mathcal{A}(\mathbb{S}_R^4)$.

Then the canonical map

$$\chi : \mathcal{A}(\mathbb{S}_R^7) \otimes_{\mathcal{A}(\mathbb{S}_R^4)} \mathcal{A}(\mathbb{S}_R^7) \rightarrow \mathcal{A}(\mathbb{S}_R^7) \otimes H, \quad \chi(p' \otimes p) = p' \delta(p)$$

is bijective.

Indeed,

$$\chi(\langle \psi | \otimes_{\mathcal{A}(\mathbb{S}_R^4)} | \psi \rangle) = \langle \psi | \delta(| \psi \rangle) = \langle \psi, \psi \rangle \otimes w = \mathbf{1} \otimes w,$$

showing surjectivity of χ . This is enough since ... (H is classical).

The next fibration

A similar (if more involved) algebra inclusion

$$\mathcal{A}(\mathbb{S}_R^8) \hookrightarrow \mathcal{A}(\mathbb{S}_R^{15})$$

(coming from an octonionic matrix $R_{\beta\mu}^{\lambda\alpha}$)

which is a \mathbb{S}^7 -bundle

Spectral geometry

coming up

thank you