Paradoxes in measure theory and their relation to other branches of mathematics

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Banach and Tarski proved that any two subsets of $\mathbb{R}^{n+1}$ or of $\mathbb{S}^n$ $(n \geq 2)$ with non-empty interiors are equivalent by finite decomposition (so-called Banach-Tarski paradox). This result strenghtens a work of Hausdorff and shows that there is no finitely additive isometry invariant measure defined on all subsets of $\mathbb{R}^{n+1}$ or of $\mathbb{S}^n$ $(n \geq 2)$ normalizing some compact set with non-empty interior.

Applying an observation of von Neumann that Banach-Tarski like paradoxes depend on existence of free groups acting on a space in question, Tomkowicz [4] proved a very general construction that unifies the existing paradoxes in some classical spaces. The proof relies on a graph theoretical criterion proved by Laczkovich [1] and a notion of smallness introduced in Mycielski and Tomkowicz [2]. We will present the proof.

Then we will relate the concepts appearing in the proof to some results obtained only recently. We will discuss a relation of the Banach-Tarski paradox theoretical framework to the universe of set theory called $L(\mathbb{R})$ which is a model of the theory ZF + DC + AD (assuming some large cardinal numbers). The investigation was proceeded by the first time by Mycielski and Tomkowicz [3]. Finally we will discuss a partial solution to the Banach-Ruziewicz problem for Borel sets which asks if the Lebesgue measure in $\mathbb{R}^{n+1}$ or in $\mathbb{S}^n$ $(n \geq 2)$ is the only one finitely additive isometry invariant measure normalizing unit cube (or the sphere) that is defined on bounded Borel sets in the spaces in question.

Several open problems appearing in the quoted papers and also in [5]

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will be stated and discussed.

References

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