A question of Berlinkov

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Let G be a group and $G_1, \ldots, G_n, \ldots$ be a sequence of subgroups of G. Let

$$D_n = \bigcap_{k \geq n} G_k \text{ and } S_n = \langle \bigcup_{k \geq n} G_k \rangle.$$

We see

$$D_1 \subseteq \ldots \subseteq D_n \subseteq \ldots\ldots \subseteq S_n \subseteq \ldots \subseteq S_1.$$

Let us define two limits of $\{G_i : i \geq 1\}$:

$$\liminf G_n = \bigcup_n D_n \quad \text{and} \quad \limsup G_n = \bigcap_n S_n.$$

For any infinite $X \subseteq \mathbb{N}$ and appropriate limits of the X-subsequence,

$$\liminf G_n \subseteq \liminf{}_X G_n \subseteq \limsup{}_X G_n \subseteq \limsup G_n.$$

A sequences $\{G_n\}$ is called **reduced** if for every infinite $X \subseteq \mathbb{N}$ we have

$$\liminf G_n = \liminf{}_{n \in X} G_n \quad \text{and} \quad \limsup G_n = \limsup{}_{n \in X} G_n.$$

M.L.Berlinkov has proved in
"Groups with a compact structure of subgroups", Mat.Sb.34 (1954), 473 - 498, that (a) if G is countable then every infinite sequence of subgroups of G contains a reduced subsequence (Theorem 14) and (b) if $|G| \geq 2^{\aleph_0}$ and G contains a minimal generating subset, then G has an infinite sequence of subgroups without reduced subsequences (Theorem 15).

He formulated the following question: *Let G be an infinite group such that each infinite sequence of subgroups of G contains a reduced subsequence. Is G countable?*

It is easy to verify that the Jónsson group M of S.Shelah witnesses that the answer is negative. Thus the following question is justified.

> Find the minimal cardinal number κ with the property that there is a group G of cardinality κ having an ω-sequence of subgroups without reduced subsequences.

Using the approach of infinite dimensional Ramsey theory we prove that this number is $min(\mathfrak{s}, \mathfrak{b})$. Then we study this question for locally countable closure operators in general, for example operators of generation of subfields or disintegrated locally countable operators.