LARGE AND DISCRETE SETS IN GROUPS, EXTREMALLY DISCONNECTED GROUPS, FILTERS, AND ULTRAFILTERS

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Various notions of large sets in groups and semigroups naturally arise in dynamics and combinatorial number theory. Most familiar are those of syndetic, thick (or replete), and piecewise syndetic sets. (In $\mathbb{Z}$, syndetic sets are those with bounded gaps, thick sets contain arbitrarily long intervals, and piecewise syndetic sets are the intersections of syndetic sets with thick ones.) It is hard to say which is more interesting, these sets themselves or the interplay between them. Thus, piecewise syndetic sets in $\mathbb{N}$ are partition regular, contain arbitrarily long arithmetic progressions, and are characterized in terms of ultrafilters on $\mathbb{N}$ containing them; the difference set of a syndetic set in a countable Abelian group almost (up to a set of upper Banach density zero) contains a set open in the Bohr topology; the quotient set of any piecewise syndetic set in any semigroup contains the quotient set of a syndetic set, and so on.

Large sets find numerous applications to dynamics, Ramsey theory, the ultrafilter semigroup on $\mathbb{N}$, the Bohr compactification, and so on. Quite recently, Evgenii Reznichenko and the author have found yet another application of large sets. Namely, we introduced special large sets in groups, which we called fat sets, and applied them to construct a discrete set with precisely one limit point in any countable nondiscrete topological group in which the identity element has nonrapid filter of neighborhoods; using this technique, we proved, in particular, the nonexistence of a countable nondiscrete extremally disconnected group in ZFC.

Fat sets and their relationship to other large sets in groups will be discussed. Special emphasis will be placed on the construction of discrete sets with one limit point (by using fat sets) and its consequences, including the nonexistence in ZFC of a countable nondiscrete extremally disconnected group; other extremal properties of topological groups will also be touched upon.

A fundamentally different construction of discrete sets in Boolean groups will also be presented; in the countable case, it yields a closed discrete basis.

Finally, various kinds of filters and ultrafilters on ω will be considered in relation to countable extremally disconnected groups. In particular, analyzing the topology of free Boolean topological groups of filter spaces and using certain facts concerning the extremal disconnectedness of such groups, we will show that if a filter has any of the well-known selectivity-type properties (that is, properties equivalent to being selective for ultrafilters), then this filter is an ultrafilter.

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