Reflection Theorems on non-existence of orthonormal bases

Sakaé Fuchino (渕野 昌)

May 31, 2017

Abstract

For the scalar field $K = \mathbb{R}$ or $\mathbb{C}$, an inner-product space over $K$ is also called a pre-Hilbert space (over $K$). An orthonormal system $B \subseteq X$ is called an orthonormal basis if $B$ spans a dense subspace of $X$. Under the lack of completeness and separability of the space, a maximal orthonormal system $B$ of $X$ need not to be an orthonormal basis. Halmos proved in 1970s that there are even pre-Hilbert spaces without any orthonormal bases.

We prove that both of the following reflection statements (A), (B) are equivalent to the Fodor-type Reflection Principle (FRP).

(A) For any regular $\kappa > \omega_1$ and any dense subspace $X$ of $\ell_2(\kappa)$ (with the inner product induced from the inner product of $\ell_2(\kappa)$), if $X$ does not have any orthonormal basis then

$$S_X = \{ \alpha < \kappa : X \downarrow \alpha \text{ d.n. have any orthonormal basis} \}$$

is stationary in $\kappa$.

(B) For any regular $\kappa > \omega_1$ and any dense subspace $X$ of $\ell_2(\kappa)$, if $X$ does not have any orthonormal basis then

$$S_{\ell_2(\kappa)} = \{ U \in [\kappa]^{\omega_1} : X \downarrow U \text{ d.n. have any orthonormal basis} \}$$

is stationary in $[\kappa]^{\omega_1}$.

With this equivalence, both of the statements (A) and (B) are also shown to be equivalent to many topological reflection statements already known to be equivalent to FRP: including:

(C) A locally compact regular space $X$ is non-metrizable if and only if there is a non-metrizable subspace $Y$ of $X$ of cardinality $\leq \aleph_1$.

(D) A $T_1$ space $X$ with point countable base is non-left-separated if and only if there is a non-left-separated subspace $Y$ of $X$ of cardinality $\leq \aleph_1$. 
Sakaé Fuchino
Kobe University
Graduate School of System Informatics
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
email: fuchino@diamond.kobe-u.ac.jp