Reflection Theorems on non-existence of orthonormal bases

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Abstract

For the scalar field $K = \mathbb{R}$ or $\mathbb{C}$, an inner-product space over K is also called a *pre-Hilbert space* (over K). An orthonormal system $B \subseteq X$ is called an *orthonormal basis* if B spans a dense subspace of X. Under the lack of completeness and separability of the space, a maximal orthonormal system B of X need not to be an orthonormal basis. Halmos proved in 1970s that there are even pre-Hilbert spaces without any orthonormal bases.

We prove that both of the following reflection statements (A), (B) are equivalent to the Fodor-type Reflection Principle (FRP).

(A) For any regular $\kappa > \omega_1$ and any dense subspace X of $\ell_2(\kappa)$ (with the inner product induced from the inner product of $\ell_2(\kappa)$), if X does not have any orthonormal basis then

$$S_X = \{\alpha < \kappa : X \downarrow \alpha \text{ d.n. have any orthonormal basis}\}$$

is stationary in κ.

(B) For any regular $\kappa > \omega_1$ and any dense subspace X of $\ell_2(\kappa)$, if X does not have any orthonormal basis then

$$S_X^{\aleph_1} = \{U \in [\kappa]^{\aleph_1} : X \downarrow U \text{ d.n. have any orthonormal basis}\}$$

is stationary in $[\kappa]^{\aleph_1}$.

With this equivalence, both of the statements (A) and (B) are also shown to be equivalent to many topological reflection statements already known to be equivalent to FRP: including:

(C) A locally compact regular space X is non-metrizable if and only if there is a non-metrizable subspace Y of X of cardinality $\leq \aleph_1$.

(D) A T_1 space X with point countable base is non-left-separated if and only if there is a non-left-separated subspace Y of X of cardinality $\leq \aleph_1$.

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