Extrinsic geometries and differential equations of type $\mathfrak{sl}_3$

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The extrinsic geometries and the geometries of systems of linear differential equations are categorically isomorphic and their elementary algebraic skeletons are represented by the pairs $(\mathfrak{g}, V)$, where $\mathfrak{g} = \bigoplus_{p \in \mathbf{Z}} \mathfrak{g}_p$ is a transitive graded Lie algebra and $V = \bigoplus_{q \in \mathbf{Z}} V_q$ is a graded $\mathfrak{g}$ -module: To each such pair $(\mathfrak{g}, V)$, there are a standard immersion to a flag variety, called Klein model, $\phi : G/G^0 \to \mathrm{Flag}(V)$ as well as a class $C(\mathfrak{g}, V)$ of immersions, called Cartan deformations, $\varphi : (M, \mathfrak{f}) \to \mathrm{Flag}(V)$ such that the first order approximation of φ is isomorphic to ϕ, where G is a Lie group with Lie algebra $\mathfrak{g}$, G^0 a closed Lie subgroup of G with Lie algebra $\mathfrak{g}^0 = \bigoplus_{p \geq 0} \mathfrak{g}_p$, and $(M, \mathfrak{f})$ a filtered manifold of type $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$. One of the fundamental problems of extrinsic geometries is to determine whether any given two immersions φ_1 and $\varphi_2 \in C(\mathfrak{g}, V)$ are equivalent or not. By a joint work with B. Doubrov and Y. Machida we have established a general method to construct the complete invariants χ of $\varphi \in C(\mathfrak{g}, V)$.

I this talk we consider as a typical example a special pair $(\mathfrak{g}, V)$ where $\mathfrak{g} = \mathfrak{sl}_3$ with the contact grading $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$, namely $\mathfrak{g}^0$ is a Borel subalgebra, and $V = \mathfrak{g}$ with the adjoint representation. We will see how the general construction goes concretely in this example. We will show moreover that we can carry out much more detailed studies on this example by the aid of some geometric consideration, representation theory, and Maple computation. In particular, we give an explicit formula of the invariant $\chi = \sum_{i \geq 1} \chi_i$ and the differential equations that the invariants χ satisfies (a sort of the Schwarzian derivative). Based on this formula, we give the classification of homogeneous immersions and systems of linear partial differential equations of this type. We treat also twistor problems and generalized versions of Gauss' theorem egregium.