TU Graz

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COMPUTATIONAL TOPOLOGY

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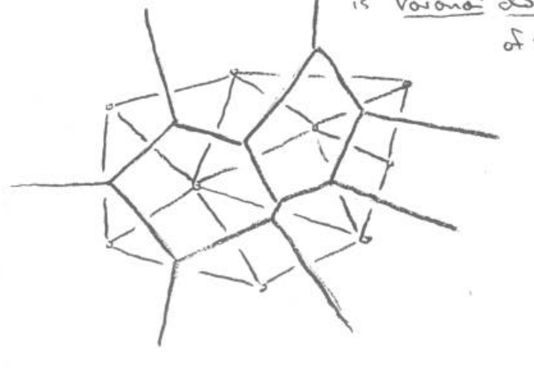
A. DELAUNAY COMPLEXES

A.1 Voronoi tessellations in $\mathbb{R}^2$

(Descartes 16..)

$X \subseteq \mathbb{R}^2$, finite

$Vor(x) = \{a \in \mathbb{R}^2 \mid \|a-x\| \le \|a-y\| \; \forall y \in X\}$ is Voronoi domain of $x \in X$.



1. $Vor(x)$ is intersection of $|X|-1$ closed half-spaces.
2. Voronoi domains have disjoint interiors and intersect in shared faces.

Voronoi tessellation is $\{Vor(x) \mid x \in X\}$.

A.2 Delaunay triangulations in $\mathbb{R}^2$

(Delone 1937)

Draw edge xy iff $Vor(x) \cap Vor(y) \neq \emptyset$
Draw triangle xyz iff
$$Vor(x) \cap Vor(y) \cap Vor(z) \neq \emptyset$$

1. Edges form a planar graph.
$\Rightarrow$ #edges $< 3|X|$,
$\Rightarrow$ #triangles $< 2|X|$.

Proof: $v - e + f = 2$ (Euler)

Suppose we add edges outside to triangulate also the outside. Then

$$3f = 2e.$$

Hence

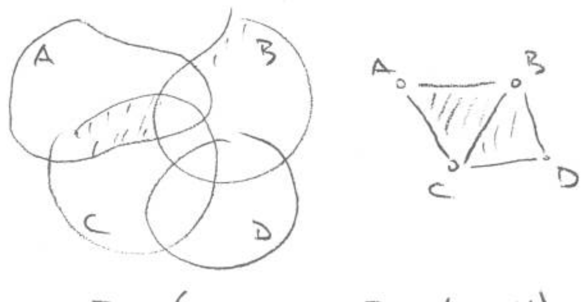
$$v - e + \frac{2e}{3} = 2 \Rightarrow e = 3v-6,$$
$$v - \frac{3f}{2} + f = 2 \Rightarrow f = 2v-4. \quad \square$$

2. $Q \in Del\,X$ iff Q has an empty circumcircle.

A.3 Nerves and Nerve Theorem.

S is collection of sets.
The *nerve* of S is the system of subcollections with non-empty intersection (Alexandrov 19..)

$$Nrv\,S = \{Q \subseteq S \mid \bigcap Q \neq \emptyset\}.$$



NERVE THM (Leray 1945, Borsuk 1948).
If S is finite and every non-empty common intersection is contractible, then $Nrv\,S \simeq \bigcup S$.

E.g. if sets in S are convex, then the non-empty common intersections are convex and therefore contractible.

A.4 Abstract simplicial complexes.

set = set
collection = set of sets
system = set of sets of sets

Let U be collection of sets.
A subcollection of U is an abstract simplex.
A system of abstract simplices, $\mathcal{A}$, is an abstract simplicial complex if $\alpha \in \mathcal{A}$ and $\beta \subseteq \alpha$ implies $\beta \in \mathcal{A}$.

1. The nerve of a collection of sets is an abstract simplicial complex.
2. The Delaunay triangulation is the nerve of the Voronoi tessellation (or is it not?)

A.5 Geometric realization

Mapping every set in U to a point in $\mathbb{R}^n$, we draw every abstract simplex as the convex hull of its points. If we avoid "improper intersections" we get a

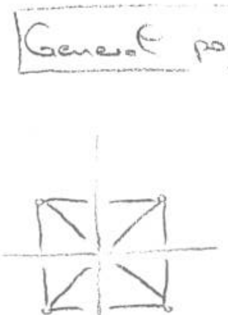
(geometric) simplicial complex: a set of simplices, K, in $\mathbb{R}^n$ such that

(i) $Q \in K$ and $P \subseteq Q$ implies $P \in K$,

(ii) $P \cap Q$ is either empty or a common face.

1. Delaunay triangulation is geometric realization of nerve of Voronoi tessellation, mapping $\mathrm{Vor}(x)$ to x (or is it not?)
2. Every a.s.c. $\mathcal{A}$ of dimension k has a geom. realization in $\mathbb{R}^{2k+1}$.
3. Underlying space, $|K|$, is union of simplices with induced topology.

A.6 General position



Nrv Vor X is a tetrahedron $\not\subset \mathbb{R}^2$.

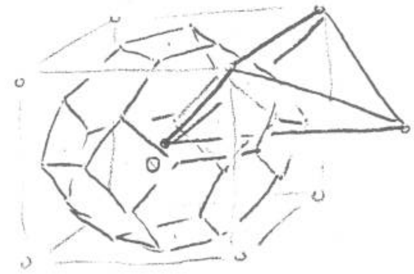
Cope:
1. Don't insist on geom. real.
2. Draw quadrangle instead.
3. Assume general position.

Def. $X \subseteq \mathbb{R}^n$ finite is in general position if no $n+2$ points belong to common n-plane and no $n+3$ points belong to common n-sphere.

(custommade for Del. tri.)

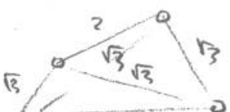
A.7 BCC lattice

The body centered cubic lattice is set of points $x = (x_1, x_2, x_3)$ with either three even or three odd integer coordinates.



All Voronoi domains are translates of $\mathrm{Vor}(0)$ = truncated octahedron

All Delaunay tetrahedra are congruent copies of

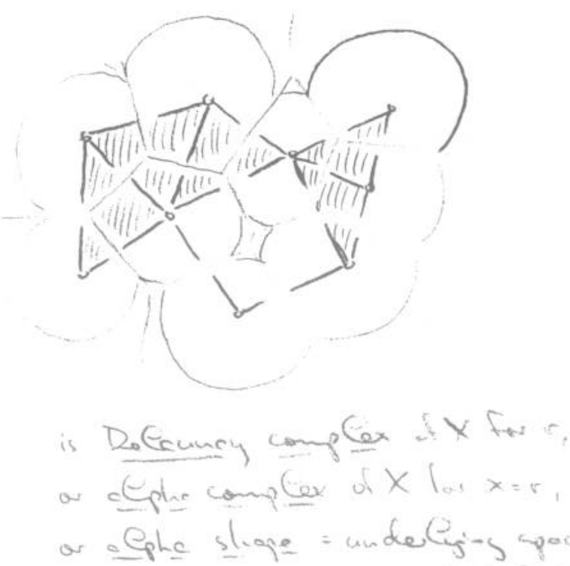


A.8 Alpha shapes

$B_r(x) = \{a \in \mathbb{R}^n \mid \|a - x\| \leq r\}$

$\mathrm{Del}_r X$ = geometric realization of Nerve $\{\mathrm{Vor}(x) \cap B_r(x) \mid x \in X\}$

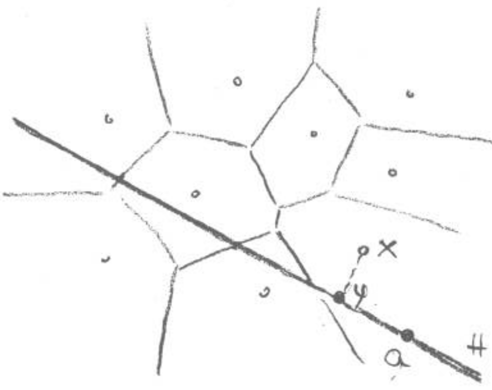
(E, Kirkpatrick, Seidel 1983)



is Delaunay complex of X for r, or alpha complex of X for $x = r$, or alpha shape = underlying space of $\mathrm{Del}_r X$.

A.9 Slices

$X \subseteq \mathbb{R}^n$ finite, H = k-plane in $\mathbb{R}^n$
Consider $\{ \mathrm{Vor}(x) \cap H \mid x \in X \}$



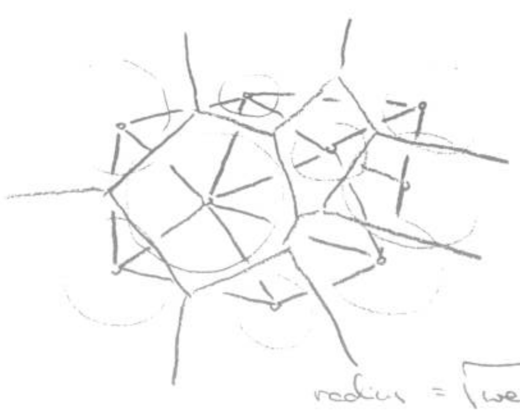
y is orthogonal proj. of x onto H.
$\|a - x\|^2 = \|a - y\|^2 + \|y - x\|^2$

A.10 Weighted Voronoi tessellation

$Y \subseteq \mathbb{R}^k \times \mathbb{R}$ finite
$\quad \overset{\Psi}{y} \quad \overset{\Psi}{\omega}$ is weight

$\pi_y(a) = \|y - a\|^2 - \omega$ is
power distance of a from
weighted point $(y, \omega) \in Y$

$\mathrm{Vor}(y) = \{ a \in \mathbb{R}^k \mid \pi_y(a) \leq \pi_x(a) \ \forall (x, \omega_x) \in Y \}$.



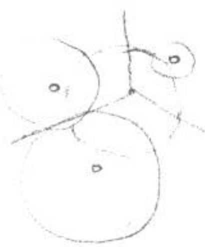
radius = $\sqrt{\text{weight}}$

A.11 Weighted Delaunay triangulation

$\mathrm{Del}\,Y$ = geometric realization of
nerve of Vor Y

1. Vertices of $\mathrm{Del}\,Y$ form subset of weighted points in Y
2. $Q \in \mathrm{Del}\,Y$ iff Q has an "empty" orthosphere:

further than orthogonal from all other w.pts.

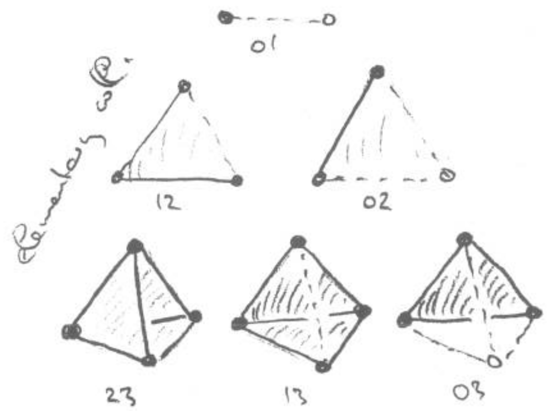


B. Discrete Morse Theory

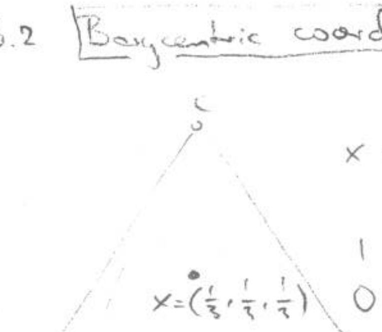
B.1 Collapses

<u>Free</u> <u>simplex</u> in K is proper face of exactly one maximal simplex. <u>Collapse</u> removes

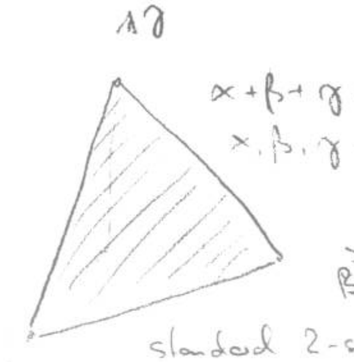
1. free simplex
2. max. simplex
3. everybody in between



B.2 Barycentric coordinates



$x = \alpha A + \beta B + \gamma C$
with
$1 = \alpha + \beta + \gamma$
$0 \leq \alpha, \beta, \gamma$



$\alpha + \beta + \gamma = 1$
$\alpha, \beta, \gamma \geq 0$

standard 2-simplex

B.3 Deformation retraction

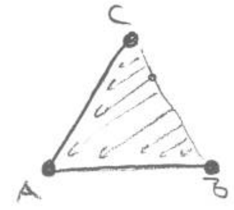
Continuous $D: \mathbb{X} \times [0,1] \to \mathbb{X}$ is <u>deformation</u> <u>retraction</u> from $\mathbb{X}$ to $\mathbb{Y} \subseteq \mathbb{X}$ if

(i) $D(x, 0) = x \quad \forall x \in \mathbb{X}$

(ii) $D(x, 1) \in \mathbb{Y} \quad \forall x \in \mathbb{X}$

(iii) $D(y, t) = y \quad \forall y \in \mathbb{Y}, t \in [0,1]$

Then $\mathbb{X}$ and $\mathbb{Y}$ are hom. equivalent.



$\mathbb{X}$ = triangle
$x = \alpha A + \beta B + \gamma C$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \to \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + c \begin{bmatrix} 1 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$c = \min\{1-\alpha, 2\beta, 2\gamma\} = \min\{2\beta, 2\gamma\}$$

$$D(x) = (1-t)\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + t \begin{bmatrix} \alpha + c \\ \beta - c/2 \\ \gamma - c/2 \end{bmatrix}.$$

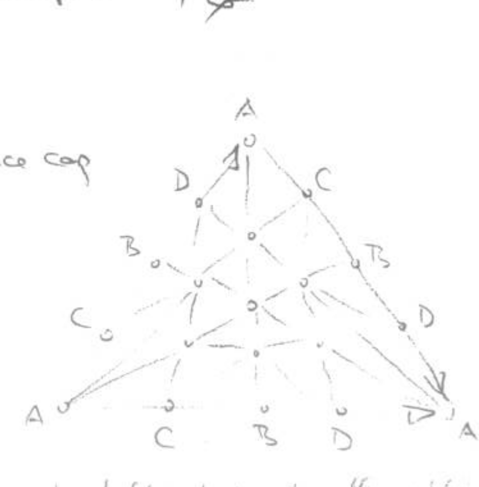
B.4 Collapsibility and contractibility

Simplicial complex K is <u>collapsible</u> if $K \searrow$ pt.

K is <u>contractible</u> if hom.eq. to point.

$$\text{Collapsibility} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{Contractibility}$$

Dunce cap



contractible but not collapsible.

B.5 Discrete Morse functions

K is simplicial complex.

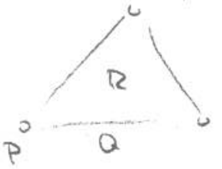
$f: K \to \mathbb{R}$ is discrete Morse function if

1. $f(P) \ge f(Q)$ for at most one face P of Q,
2. $f(R) \le f(Q)$ for at most one coface R of Q.

(Forman 1998)

We cannot have

$f(P) \ge f(Q) \ge f(R)$

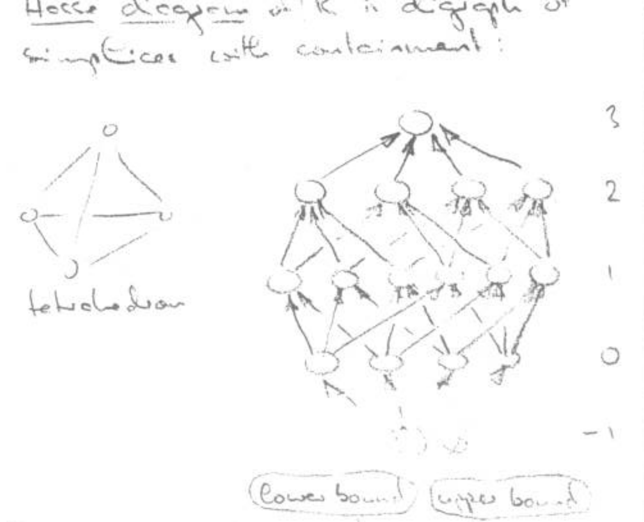


Matching formed by exceptional pairs (collapses), and unmatched simplices are critical.

Thm: $f: K \to \mathbb{R}$ is d.M.function, and $K_0 \subseteq K$ s.t. $K \setminus K_0$ contains no critical simplices. Then $K \searrow K_0$.

B.6 Hasse diagrams and intervals

Hasse diagram of K is digraph of simplices with containment:



for simplices $L \subseteq U$,
$[L, U] = \{Q \in K \mid L \subseteq Q \subseteq U\}$ is interval.

1. $|[L,U]| = 2^{\dim U - \dim L}$
2. Interval can be partitioned into pairs.

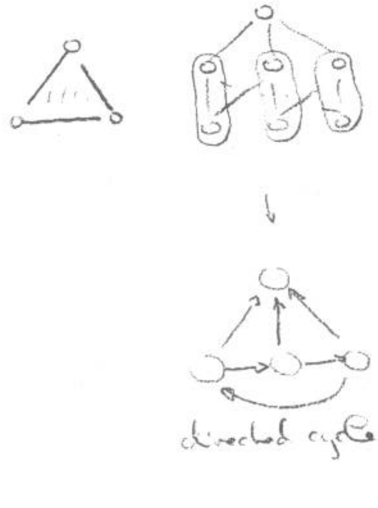
B.7 Gen. discrete vector fields

K is simplicial complex

Gen. discrete vector field, V, is partition of K into intervals.

Csay (200.), Freij (201.)

Contract intervals to nodes and keep all di-edges in Hasse diagram.



B.8 Gen. discrete gradient

... is acyclic g.d.v.f.

To prove acyclicity of V find function $f: K \to \mathbb{R}$ such that $f(P) \le f(Q)$ whenever $P \subseteq Q$, with equality iff P, Q in same interval.

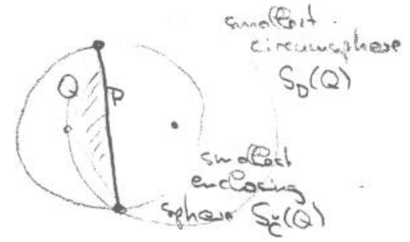
Then f is generalized discrete Morse function and V is its generalized discrete gradient.

1. V can be refined to a matching
2. f can be perturbed to a discrete Morse function.

B.9 Radius function

$\rho : \mathrm{Del}\,X \to \mathbb{R}$ defined by
$\rho(Q) = \min \{ r \mid Q \in \mathrm{Del}_r X \}$.

$\Rightarrow \mathrm{Del}_r(X) = \rho^{-1}[0,r]$.



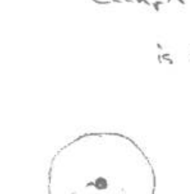
1. $Q \in \mathrm{Del}\,X$ is critical simplex of ρ if $S_D(Q) = S_{\check{C}}(Q)$.
2. $Q \in \mathrm{Del}\,X$ is upper bound of interval if $S_D(Q)$ is empty.
3. $P \subseteq Q$ belongs to same interval if all facets $P \subseteq F \subseteq Q$ are visible from center of $S_D(Q)$.

Hence, ρ is a generalized discrete Morse function.

B.10 Čech complexes

$X \subseteq \mathbb{R}^n$ finite, $B_r(x)$ is ball of radius $r \geq 0$ centered at $x \in X$.

$\check{\mathrm{C}}\mathrm{ech}_r X = \{ Q \subseteq X \mid \bigcap_{x \in Q} B_r(x) \neq \emptyset \}$
is Čech complex of X for radius r.



1. $Q \in \check{\mathrm{C}}\mathrm{ech}_r(X)$ iff radius of $S_{\check{C}}(Q)$ is at most r.

simplex spanned by X

$\rho_{\check{C}} : \Delta(X) \to \mathbb{R}$ defined by
$\rho_{\check{C}}(Q) = $ radius of $S_{\check{C}}(Q)$.

$\Rightarrow \check{\mathrm{C}}\mathrm{ech}_r X = \rho_{\check{C}}^{-1}[0,r]$.

2. $Q \in \Delta(X)$ is lower bound of interval if $S_{\check{C}}(Q) = S_D(Q)$. Corr. upper bound is points inside and on $S_{\check{C}}(Q)$.
3. Q is critical simplex of $\rho_{\check{C}}$ if $S_{\check{C}}(Q) \cap S_D(Q)$ is empty.

B.11 Delaunay-Čech complexes

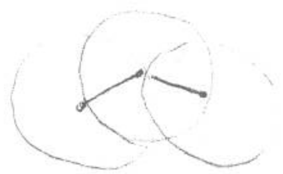
$\mathrm{Del}\check{\mathrm{C}}\mathrm{ech}_r X = \mathrm{Del}\,X \cap \check{\mathrm{C}}\mathrm{ech}_r X$.

1. $\mathrm{Del}_r X \subseteq \mathrm{Del}\check{\mathrm{C}}\mathrm{ech}_r X$ but they are not necessarily the same.

   $Q \in \mathrm{Del}_r X$ if $\bigcap_{x \in Q} [\mathrm{Vor}(x) \cap B_r(x)] \neq \emptyset$

   $Q \in \mathrm{Del}\check{\mathrm{C}}\mathrm{ech}_r X$ if

   $\bigcap_{x \in Q} \mathrm{Vor}(x) \neq \emptyset$ and $\bigcap_{x \in Q} B_r(x) \neq \emptyset$



2. Delaunay-Čech complex does not grow in intervals.

?

B.12 Collapsing Hierarchy

$X \subseteq \mathbb{R}^n$ finite, $r \geq 0$. Then

$\check{\mathrm{C}}\mathrm{ech}_r X \searrow \mathrm{Del}\check{\mathrm{C}}\mathrm{ech}_r X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X$
(Bauer, E 2015)

The Wrap complex (for $r = \infty$) was used by Geomagic for surface reconstruction.

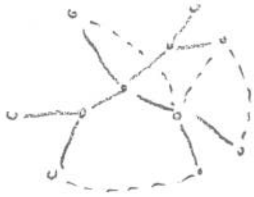
Hasse diagram $\to$ poset of intervals

$\mathrm{crit}_r(\rho) = $ set of crit. simplices Q of ρ with $\rho(Q) \leq r$.

$\mathrm{Wrap}_r X$ is lower set of $\mathrm{crit}_r(\rho)$.

C.1 Cyclomatic number

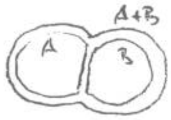
G is simple finite graph,
G is tree if connected and acyclic.

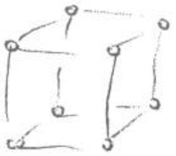


v vertices
$e = v-1$ edges

Every additional edge forms a new cycle.

Assuming G is connected, cyclomatic number is $e-(v-1)$.

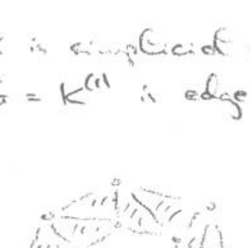




cycl. # = 5 because every cycle is lin. comb. of 5 basis cycles.

C.2 First Betti number

K is simplicial complex,
$G = K^{(1)}$ is edge skeleton.



2-chain is set of triangles, d.
boundary, ∂d, is edges shared by odd # triangles.

1-cycles c, c' are homologous (same) if $\exists d$ s.t. $c + c' = \partial d$.

$\beta_1(K)$ = # classes of hom. 1-cycles.

C.3 Homology groups

K is simplicial complex

$C_p(K)$ = group of p-chains

set of p-simplices
$Q + Q = 0$

$$\partial_p c = \sum_{Q \in c} \partial_p Q$$

$Z_p(K)$ = group of p-cycles

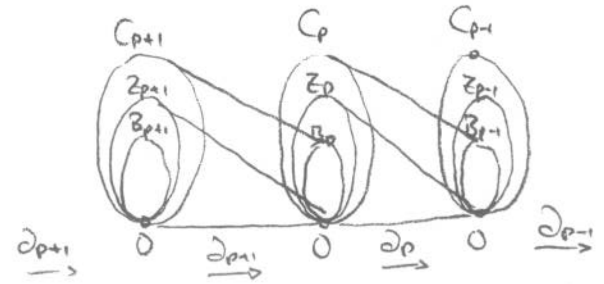
p-chains with
$\partial_p c = 0$

$B_p(K)$ = group of p-boundaries

$\exists$ (p+1)-chain d with
$\partial_{p+1} d = c$

$H_p(K) = Z_p(K) / B_p(K)$ is p-th homology group

$\beta_p(K)$ = rank $H_p(K)$ is p-th Betti number



$Z_p = \ker \partial_p$
$B_p = \operatorname{im} \partial_{p+1}$

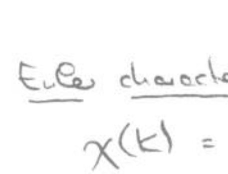
Lemma: $\partial_p \circ \partial_{p+1} = 0$.

Euler characteristic is

$$\chi(K) = \sum_{p \geq 0} (-1)^p \,\#\text{p-simplices}$$

Euler-Poincaré Formula:

$$\chi(K) = \sum_{p \geq 0} (-1)^p \beta_p(K).$$



$\beta_0 = 1$
$\beta_1 = 2$
$\beta_2 = 1$
$\chi = 1 - 2 + 1 = 0.$

C.4 Incremental Betti # algorithm

K is simplicial complex.

Sort simplices s.t. faces precede
each simplex: $Q_1, Q_2, \ldots, Q_m$;

for $p \geq 0$ **do** $\beta_p = 0$ **endfor**; $K_0 = \emptyset$;

for $i = 1$ **to** m **do**

  $K_i = K_{i-1} \cup \{Q_i\}$; $p = \dim Q_i$;

  if Q_i belongs to p-cycle in K_i

    then $\beta_p = \beta_p + 1$

    else $\beta_{p-1} = \beta_{p-1} - 1$

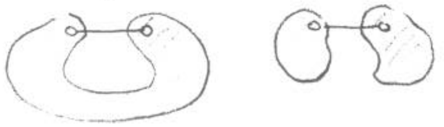
  endif

endfor.

(E, Delfinado 1995)

time = $m \log m$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In $\mathbb{R}^2$ only choice is



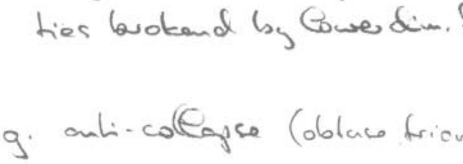
C.5 Almost collapses

$K = \mathrm{Del}\, X$,

$\varrho(Q_1) \leq \varrho(Q_2) \leq \ldots \leq \varrho(Q_m)$

ties broken by lower dim. first.

E.g. anti-collapse (obtuse triangle):



$\beta_0{++}$ $\beta_0{--}$ $\beta_1{++}$ $\beta_1{--}$

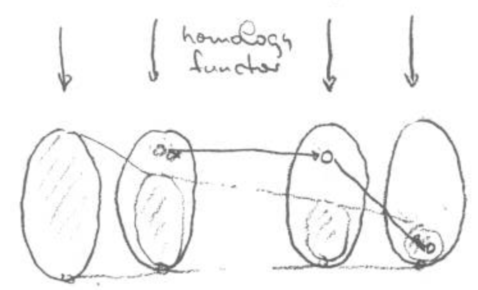
all 4 steps at same radius
cancelling effects on β #s

Similar for slightly acute triangle
except that steps for slightly diff.
radius.

C.6 Filtrations and persistence modules

Filtration is nested seq. of s.c.

$\ldots \subseteq K_{i-1} \subseteq K_i \subseteq \ldots \subseteq K_{j-1} \subseteq K_j \subseteq \ldots$



$\ldots \to H_{i-1} \to H_i \to \ldots \to H_{j-1} \to H_j \to \ldots$

$\bigoplus_{p \geq 0} H_p(K_{i-1})$

with maps
$f_i^j : H_i \longrightarrow H_j$
induced by $K_i \subseteq K_j$,
is persistence module.

C.7 Birth and death

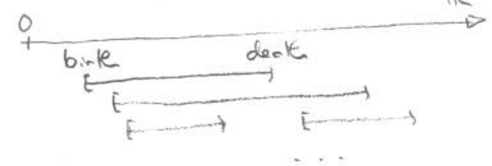
Let $\alpha \in H_i$ be a homology class.

α is born at H_i if $\alpha \notin \mathrm{im}\, f_{i-1}^i$
and it dies entering H_j if
$f_i^{j-1}(\alpha) \notin \mathrm{im}\, f_{i-1}^{j-1}$ but
$f_i^j(\alpha) \in \mathrm{im}\, f_{i-1}^j$.

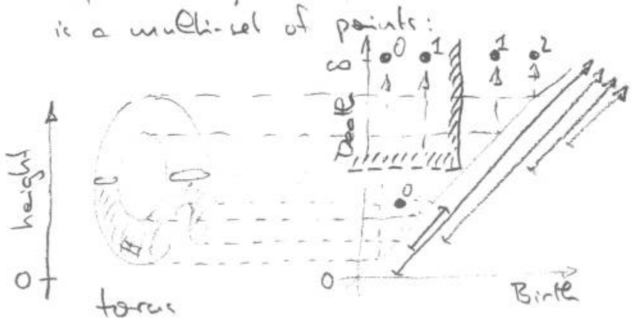
The persistence of α is $j - i$
or $\varrho(Q_j) - \varrho(Q_i)$.

C.8 Barcodes and persistence diagrams

The *barcode* of the filtration is a multi-set of intervals:



Equivalently, the *persistence diagram* is a multi-set of points:



(E, Letscher, Zomorodian 2000)

Add all diagonal points to diagram for good measure.

C.9 Bottleneck distance

Given persistence diagrams $F = \mathrm{Dgm}(f)$ and $G = \mathrm{Dgm}(g)$, the *bottleneck distance* is the length of the longest edge in the minimizing matching:

$$W_\infty(F,G) = \inf_{\mathrm{bij}: F \to G} \max_{a \in F} \| a - \mathrm{bij}(a) \|_\infty .$$

C.10 Stability

Functions $f, g : \mathbb{X} \to \mathbb{R}$ are *tame* if all sublevel sets have finite rank hom., and there are finitely many hom. crit. values.

Thm.

$$W_\infty(F,G) \le \|f-g\|_\infty .$$

(Cohen-Steiner, E, Harer 2007)

C.11 Vietoris-Rips complexes

$X \subseteq \mathbb{R}^n$ finite

$$\mathrm{Rips}_r X = \{ Q \subseteq X \mid \mathrm{diam}(Q) \le 2r \}$$

is *Vietoris-Rips* complex of X for radius $r \ge 0$.

1. 1-skeletons of $\mathrm{Rips}_r X$ and $\check{\mathrm{C}}\mathrm{ech}_r X$ are the same.
2. $\mathrm{Rips}_r X$ is clique complex of its 1-skeleton $\Rightarrow \check{\mathrm{C}}\mathrm{ech}_r X \subseteq \mathrm{Rips}_r X$
3. $\mathrm{Rips}_r X \subseteq \check{\mathrm{C}}\mathrm{ech}_{r\sqrt{2}} X$.

Map radius to $f(r) = \log_2(r+1)$

$$\log_2(\sqrt{2}r+1) < \log_2(r+1) + \tfrac{1}{2}$$

Hence

$$W_\infty(\check{\mathrm{C}}\mathrm{ech}, \mathrm{Rips}) < \tfrac{1}{2} .$$