Algebro-geometric method for explicitly solving algebraic differential equations

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Abstract

An algebraic differential equation (ADE) is given by a polynomial relation between a desired function $y(x_1, \ldots, x_n)$, and some of its derivatives. If $n = 1$ we speak of an algebraic ordinary differential equation (AODE), otherwise of an algebraic partial differential equation (APDE). We are interested in finding prescribed types of solutions, e.g., rational or algebraic solutions, to such ADEs. Indeed, if possible, we determine so-called general solutions, which are whole multiparametric families of solutions. The problem of determining general solutions of first-order AODEs can be dated back to the work of L. Fuchs and H. Poincaré.

In the algebro-geometric method we associate to a given (system of) ADE(s) an algebraic variety. From the rational components of this variety we determine explicit rational solutions of the differential problem; indeed so-called rational general solutions, i.e., full families of rational solutions. As an example consider the AODE $F(x, y, y') \equiv y'^2 + 3y' - 2y - 3x = 0$. A rational general solution is $y(x) = \frac{1}{2}(x^2 + 2cx + c^2 + 3c)$.

The basic idea of this approach was described by Feng and Gao in [1] and [2] for the situation of an autonomous AODE of order 1, $F(y, y') = 0$, F an irreducible bivariate polynomial over $\mathbb{Q}$. If the AODE has a rational solution $y(x)$, then $(y(x), y'(x))$ is a rational parametrization of the so-called associated curve $\mathcal{C} : F(u, v) = 0$. Indeed, as they prove, this is a proper 1-1 parametrization. All proper parametrizations are related by linear rational transformations. So we need to decide whether we can find a transformation which leads to a parametrization, in which the second component is the derivative of the first. This can be done algorithmically, leading to a rational solution of the AODE or to the answer that no such rational solution exists.

The method has been extended to non-autonomous AODEs in [4], and recently to algebraic partial differential equations in [3].

References

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