On q-deformations of the Heun equation

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Heun's differential equation is a standard form of the second order linear differential equation with four regular singularities on the Riemann sphere, and it is written as

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-t)}y = 0,$$

with the condition $\gamma + \delta + \epsilon = \alpha + \beta + 1$.

In [2], q-difference deformations of Heun's differential equation were obtained by degenerations of Ruijsenaars-van Diejen operators, which were also obtained in connection with q-Painlevé equations. One of the q-deformation is the q-Heun equation written as

$$\{a_0 + a_1 x + a_2 x^2\}g(x/q) + \{b_0 + b_1 x + b_2 x^2\}g(x) + \{c_0 + c_1 x + c_2 x^2\}g(qx) = 0. \qquad (1)$$

where $a_0 a_2 c_0 c_2 \neq 0$. The equation (1) was discovered by Hahn [1] in 1971.

The q-Heun equation was obtained in [2] as an eigenfunction of the fourth degeneration of the Ruijsenaars-van Diejen operator of one variable

$$\begin{aligned} A^{\langle 4\rangle} = {} & x^{-1}(x - q^{h_1+1/2}t_1)(x - q^{h_2+1/2}t_2)T_{q^{-1}} + q^{\alpha_1+\alpha_2}x^{-1}(x - q^{l_1-1/2}t_1)(x - q^{l_2-1/2}t_2)T_q \\ & - \{(q^{\alpha_1} + q^{\alpha_2})x + q^{(h_1+h_2+l_1+l_2+\alpha_1+\alpha_2)/2}(q^{\beta/2} + q^{-\beta/2})t_1 t_2 x^{-1}\} \end{aligned}$$

with eigenvalue E, where $T_{q^{-1}}g(x) = g(x/q)$ and $T_q g(x) = g(qx)$. Namely the equation (1) is written as $A^{\langle 4\rangle}g(x) = Eg(x)$.

We can obtain variants of the q-Heun equation as eigenfunctions of the third or second degeneration of the Ruijsenaars-van Diejen operator of one variable. Then we investigate local properties of the q-Heun equation and its variants. In particular we characterize the variants of the q-Heun equation by using analysis of regular singularities [3]. We also consider the quasi-exact solvability of the q-Heun equation and its variants. Namely we investigate finite-dimensional subspaces which are invariant under the action of the q-Heun operator or variants of the q-Heun operator [3].

References

[1] W. Hahn, On linear geometric difference equations with accessory parameters, *Funkcial. Ekvac.* **14** (1971), 73–78.

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