# The Riemann-Hilbert problem and singularity formation in the localized induction approximation

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We are concern with the differential equation of the form

$$z_t = -k_s n - \frac{1}{2}k^2 T, \quad t, s \in \mathbb{R}, \tag{1}$$

where $z$ is the flow of regular curves living in the complex plane, $s$ is the arc-length parameter, $T$ is the field of tangent vectors, $n := iT$ is the oriented normal vector field and $k$ is the curvature defined by $T_s = kn$. The equation (1) is called the *localized induction approximation* and can be considered as the geometric flow, whose evolution is similar to the contour dynamics of a vortex patch subjected to the 2D Euler equation (see e.g. [3]). Our aim is to show the existence of a regular family of self-similar solutions for (1), which develops a spiral singularity at finite time. To be more precise, we prove that for any $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mu \in \mathbb{R}$, there are $\theta^+, \theta^- \in [0, 2\pi)$ and a self-similar solution $z^{\mu,a}$ of the equation (1), such that $\theta^+ - \theta^- = 2a$ and $\lim_{t\to 0} z^{\mu,a}(t) = z_0^{\mu,a}$ in the space of tempered distributions, where

$$z_0^{\mu,a}(s) = \frac{s}{\sqrt{1+4\mu^2}} e^{i(\theta^- - 2\mu \log|s|)} \chi_{(-\infty,0)}(s) + \frac{s}{\sqrt{1+4\mu^2}} e^{i(\theta^+ - 2\mu \log s)} \chi_{(0,+\infty)}(s), \quad s \in \mathbb{R}. \tag{2}$$

Furthermore we show that the curvature flow $k^{\mu,a}$ corresponding to the solution $z^{\mu,a}$ satisfies the modified Korteweg-de Vries (mKdV) equation

$$k_t + k_{sss} + \frac{3}{2}k^2 k_s = 0 \tag{3}$$

with the following initial condition

$$k^{\mu,a}(s,0) = a\,\delta(s) + \mu\,\mathrm{p.v.}\,(1/s), \tag{4}$$

where $\delta(s)$ is the Dirac delta function and $\mathrm{p.v.}\,(1/s)$ is the Cauchy principal value. The function (2) is called a *logarithmic spiral* and plays a crucial role in the fluid mechanics and turbulence modeling. The method of the proof is searching for the solutions $z^{\mu,a}$ in a class of self-similar functions, whose profiles are purely imaginary solutions of the second Painlevé (PII) equation

$$u''(x) = xu(x) + 2u^3(x) - \alpha \tag{5}$$

with $\alpha := -i\mu$. Following [2], each solution of the (PII) equation is determined by a Riemann-Hilbert problem whose jump function is expressed by Stokes multipliers $(s_1, s_2, s_3) \in \mathbb{C}^3$ satisfying the constraint

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = -2\sin(\pi\alpha).$$

After improving the integral formulas from [1], we find $m \in i\mathbb{R}$ such that the purely imaginary Ablowitz-Segur solution corresponding to the following choice of the Stokes data

$$s_1 = -\sin(i\pi\mu) - im, \quad s_2 = 0, \quad s_3 = -\sin(i\pi\mu) + im, \tag{6}$$

is an appropriate profile in the construction of the solution $z^{\mu,a}$. The obtained results are effect of the joint work with L. Vega [4] and improves those from [5], where the authors use the perturbation argument together with the methods of ODEs to prove the existence of solutions $z^{\mu,a}$ for the parameters $\mu$ and $a$ that are sufficiently close to zero.

## References


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