Complex differential equations with movable algebraic singularities

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Abstract

A classical question in the theory of ordinary differential equations in the complex plane is to determine what types of movable singularities (i.e. those singularities depending on the initial data of the equation) a local, analytic solution can develop when analytically continued in the complex plane. For certain classes of second-order differential equations and two-dimensional Hamiltonian systems of equations it is shown that under certain mild conditions, the only movable singularities that can occur in a solution are of algebraic type, represented by Puiseux series with finite principle parts [1]. In the special case that all movable singularities are poles one obtains instances of the Painlevé equations, certain integrable non-linear second-order differential equations which possess a rich mathematical structure and the solutions of which, the so-called Painlevé transcendents, are also referred to as non-linear special functions. A Hamiltonian system of this type is also studied in [2]. One particularly interesting aspect of the Painlevé equations is the so-called Okamoto's space of initial conditions, which is obtained by compactifying the phase space on which the original equation is defined to some rational surface, followed by a number of blow-ups, a construction to remove certain points at which the equation is of indeterminate form. In the case of the Painlevé equations, the space is uniformly foliated by the solutions of the equation and around every point in this space the equation is defined as a regular initial value problem, see also [3] for the system introduced in [2].

The concept of the space of initial conditions can be extended to certain classes of differential equations with algebraic singularities mentioned above, leading to an algorithmic procedure by which to determine, for a given equation, what different types of movable singularities can occur in the solutions of the equations in the complex plane. This part of the talk is joint work with Galina Filipuk.

References

[1] T. Kecker (2016), Polynomial Hamiltonian systems with movable algebraic singularities, *Journal d'Analyse Mathématique* 129, 197–218

[2] T. Kecker (2016), A cubic Hamiltonian system with meromorphic solutions, *Computational Methods and Function Theory* 16, 307–317

[3] T. Kecker (2018), Space of initial conditions for a cubic Hamiltonian system, *Complex Variables and Elliptic Equations*, 11 pages (published online)