CURVATURE BOUNDED CONJUGATE SYMMETRIC CONNECTIONS WITH COMPLETE METRIC

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Conjugate connections naturally appeare in affine differential geometry. Namely, for a non-degenerate equiaffine hypersurface $f : M \to \mathbf{R}^{n+1}$ and its conormal map $f^*M \to (\mathbf{R}^{n+1})^*$ we have the induced connections which are conjugate relative to the second fundamental form g for f. Conjugate connections exist in the theory of Hessian manifolds and their generalizations, that is, statistical manifolds. The first publications on Hessian and statistical structures appeared in the 80-ties, see e.g. the monograph texts by A. Amari [1] and Lauritzen [2]. In [3] Nomizu and Simon, the leading experts in affine differential geometry, published a paper in which they proposed the study of conjugate connections on abstract manifolds, that is, in the situation where the conjugate connections are not necessarily those obtained on equiaffine hypersurfaces. In the 90-ties, after almost 90 years of existing and developing the theory of affine hypersurfaces, there were a lot of beautiful geometric results on the induced structures on affine hypersurfaces and a lot of examples of remarkable models, although the case of hypersurfaces is very special. The study of conjugate connections on abstract manifolds intensified after 2010. Usually the name "a statistical structure" is now used. In particular, in 2015 in [4] a sectional curvature for conjugate connections was introduced and basic properties of this curvature were established. The notion of sectional curvature is attributed to Riemannian geometry in a strong way and it is very unusual that it can be successfully extended to non-metric connections. In fact, we have two new sectional curvatures for statistical structures. The first one is defined by the curvature tensor $\mathcal{R} = \frac{R+\overline{R}}{2}$, where R and $\overline{R}$ are the curvature tensors for the conjugate connections. The second one is determined by the difference tensor of the structure. In general, Schur's lemma does not hold for any of the new sectional curvatures. It holds for conjugate symmetric statistical structures. This category of structures corresponds to the category of equiaffine spheres in the theory of affine hypersurfaces, but, even from the local view-point, it is much larger. The conjugate symmetric structures were also noticed by statisticians. In particular, the name "a conjugate symmetric statistical structure" is taken from [2]. The class of conjugate symmetric statistical structures is as important in the theory of statistical structures as the class of equiaffine spheres in affine differential geometry. Within the last class we have affine spheres which are trace-free equiaffine spheres. Again, the class of trace-free conjugate symmetric statistical structures is much larger than the class of affine spheres, even from the local view-point. For instance, the sectional curvature determined by the curvature tensor $\mathcal{R}$ on connected equiaffine spheres is constant. In [5] the following generalization of famous theorems of Blaschke, Deicke and Calabi for affine spheres are proved in the case of statistical structures on abstract manifolds of non-constant sectional curvature:

Theorem 0.1. *Let $(g, \nabla, \overline{\nabla})$ be a trace-free conjugate symmetric statistical structure on a manifold M. Assume that g is complete on M. If the sectional curvature* *determined by $\mathcal{R}$ is bounded from below and above on M then the Ricci tensor of g is bounded from below and above on M. If this sectional curvature is non-negative everywhere on M then the statistical structure is trivial, that is, ∇ is the LeviCivita connection for g. If this sectional curvature is positive and bounded from 0 by a positive constant then, additionally, M is compact and its first fundamental group is finite.*

References

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