HYPERSURFACES SATISFYING CURVATURE CONDITIONS DETERMINED BY THE OPOZDA-VERSTRAELEN AFFINE CURVATURE TENSOR

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Dedicated to Professor Udo Simon on his eigthtieth birthday

ABSTRACT

Let $\mathbb{A}^{n+1}$, $n \geq 2$, be the $(n+1)$-dimensional real affine space, i.e. $\mathbb{R}^{n+1}$ equipped with its standard flat connection $\widetilde{\nabla}$ and the volume element $\widetilde{\Theta}$ given by the determinant. Let M be a non-degenerate hypersurface in $\mathbb{A}^{n+1}$ with the affine normal ξ and the induced equiaffine structure (∇, θ) (see, e.g., [5, 7]). Using the curvature tensor R of ∇, the affine shape operator $\mathcal{S}$, the Blaschke metric h of M, the Gauss equation $R(X,Y)Z = h(Y,Z)\mathcal{S}X - h(X,Z)\mathcal{S}Y$ and the Ricci equation $h(X,\mathcal{S}Y) = h(Y,\mathcal{S}X)$ of M in $\mathbb{A}^{n+1}$ we can define the generalized curvature tensor R^* of M named the *Opozda-Verstraelen affine curvature tensor* R^* of M [5, 6]. Namely, we have

$$R^*(X_1,\ldots,X_4) = h(R(X_1,X_2)\mathcal{S}X_3,X_4) = h(h(X_2,\mathcal{S}X_3)\mathcal{S}X_1 - h(X_1,\mathcal{S}X_3)\mathcal{S}X_2,X_4)$$
$$= h(X_2,\mathcal{S}X_3)h(\mathcal{S}X_1,X_4) - h(X_1,\mathcal{S}X_3)h(\mathcal{S}X_2,X_4) = S(X_1,X_4)S(X_2,X_3) - S(X_1,X_3)S(X_2,X_4),$$

where the $(0,2)$-tensor S is defined by $S(X,Y) = h(X,\mathcal{S}Y)$ and $X,Y,Z,X_1,\ldots,X_4$ are tangent vector fields on M. We denote by $\mathrm{Ricc}(R^*)$, $\kappa(R^*)$ and $\mathrm{Weyl}(R^*)$ the Ricci tensor, the scalar curvature and the Weyl tensor determined by the metric h and the tensor R^* [5]. The non-degenerate hypersurface M in $\mathbb{A}^{n+1}$, $n \geq 3$, is said to be *affine quasi-umbilical* at a point of $x \in M$ if at x we have $\mathrm{rank}(S - \alpha\, h) = 1$, for some $\alpha \in \mathbb{R}$ [5]. In [5] it was proved that for affine quasi-umbilical M in $\mathbb{A}^{n+1}$, $n \geq 3$, the tensor $\mathrm{Weyl}(R^*)$ vanishes. The converse statement is true when $n \geq 4$ [2]. We refer to [8] for examples of affine quasi-umbilical hypersurfaces. In [1, 2, 5, 6] results on hypersurfaces M in $\mathbb{A}^{n+1}$ satisfying other conditions imposed on R^* are studied. For instance, we have $R^* \cdot R^* = Q(\mathrm{Ricc}(R^*), R^*)$ on M [2]. Using results of [3, 4] we can obtain further results on this class of hypersurfaces. In particular, if the conditions $R^* \cdot R^* = L\, Q(h, R^*)$ and $\mathrm{rank}(\mathrm{Ricc}(R^*) - \rho\, h) \geq 2$, for some functions L and ρ, are satisfied on a subset $U \subset M$, then at every point of U the tensor R^* is a linear combination of the Kulkarni-Nomizu products $\mathrm{Ricc}(R^*) \wedge \mathrm{Ricc}(R^*)$, $h \wedge \mathrm{Ricc}(R^*)$ and $h \wedge h$. This implies that at every point of U the difference tensor $\mathrm{Weyl}(R^*) \cdot R^* - R^* \cdot \mathrm{Weyl}(R^*)$ is a linear combination of the Tachibana tensors $Q(\mathrm{Ricc}(R^*), \mathrm{Weyl}(R^*))$ and $Q(h, \mathrm{Weyl}(R^*))$. Precisely, we have on U

$$\mathrm{Weyl}(R^*) \cdot R^* - R^* \cdot \mathrm{Weyl}(R^*) \;=\; Q(\mathrm{Ricc}(R^*), \mathrm{Weyl}(R^*)) - (\kappa(R^*)/(n-1))\, Q(h, \mathrm{Weyl}(R^*)).$$

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