Determinant and trace of the Second Variation

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Second Variation is the Hessian of the cost for an optimal control problem at an extremal. If the extremal is regular, then Second Variation is a quadratic form on a Hilbert space defined by a symmetric operator $I + A$ where $A$ is compact. In general, $A$ is not a trace class operator, the series $\sum_{\lambda \in \text{spec} A} |\lambda|$ diverges. We show however that the determinant of $I + A$ and the trace of $A$ can be properly defined and computed. More precisely, let $i_{\lambda}$ be the multiplicity of the eigenvalue $\lambda$. We prove that the sequences $\prod_{\lambda \in \text{spec} A, |\lambda| \geq \varepsilon} (1 + \lambda)^{i_{\lambda}}$ and $\sum_{\lambda \in \text{spec} A, |\lambda| \geq \varepsilon} i_{\lambda}\lambda$ converge as $\varepsilon \to 0$ and give explicit expressions for the limits in terms of “Jacobi fields” along the extremal. In the case of the 1-dimensional variational problem with the cost $\int_0^1 \dot{x}(\tau)^2 - \nu x(\tau)^2 \, d\tau$, we get classical Euler identities:

$$\prod_{n=1}^{\infty} \left(1 - \frac{\nu}{(\pi n)^2}\right) = \frac{\sin \sqrt{\nu}}{\sqrt{\nu}}, \quad \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2} = \frac{1}{6}.$$