Determinant and trace of the Second Variation

A. Agrachev (SISSA, Trieste)

Second Variation is the Hessian of the cost for an optimal control problem at an extremal. If the extremal is regular, then Second Variation is a quadratic form on a Hilbert space defined by a symmetric operator $I + A$ where A is compact. In general, A is not a trace class operator, the series $\sum\limits_{\lambda \in \mathrm{spec} A} |\lambda|$ diverges. We show however that the determinant of $I + A$ and the trace of A can be properly defined and computed. More precisely, let i_λ be the multiplicity of the eigenvalue λ. We prove that the sequences $\prod\limits_{\substack{\lambda \in \mathrm{spec} A \\ |\lambda| \geq \varepsilon}} (1+\lambda)^{i_\lambda}$ and $\sum\limits_{\substack{\lambda \in \mathrm{spec} A \\ |\lambda| \geq \varepsilon}} i_\lambda \lambda$ converge as $\varepsilon \to 0$ and give explicit expressions for the limits in terms of "Jacobi fields" along the extremal. In the case of the 1-dimensional variational problem with the cost $\int\limits_0^1 \dot{x}(\tau)^2 - \nu x(\tau)^2 \, d\tau$, we get classical Euler identities:

$$\prod_{n=1}^{\infty} \left(1 - \frac{\nu}{(\pi n)^2}\right) = \frac{\sin \sqrt{\nu}}{\sqrt{\nu}}, \qquad \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2} = \frac{1}{6}.$$