Study of Geodesic Flows near the Cut-Locus, for Global Tracking Control Design

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We consider the problem of designing globally-exponentially stable control laws for Lagrangian mechanical systems on compact manifolds, for tracking a bounded speed reference trajectory, using the intrinsic Riemannian structure. We first design a kinematic feedback law for the velocity of a particle evolving in a compact configuration manifold (later extended to a force feedback), by extending the well known gradient based feedback law for Euclidean spaces. The velocity control law consists of a feedback term which is the gradient of the geodesic distance, and a feed-forward term which is the parallel transport of the reference trajectory's velocity, along the minimal geodesic. We demonstrate that the closed loop system is exponentially stable, as long as the particle starts within *injectivity radius* of the reference. The primary impediment in achieving global stability is that the particle may encounter the *cut-locus* of the reference trajectory, where the control law is no longer defined. For global stability, it is necessary to show that the gradient based control law never encounters the cut locus of the reference trajectory, as long as it starts outside the cut-locus, which is of zero measure.

We start first by analyzing the behaviour of uniformly perturbed minimal geodesics, to a point p, near the cut-locus C_p. We show that it is possible to construct a neighborhood T_p of C_p, of arbitrarily small thickness, uniformly given for any generic p, such that minimal geodesics to p starting from any point on the boundary ∂T_p, strictly emerge out of the neighborhood, in the angle sense. Further, for a given thickness of T_p, we show that this angle can be uniformly lower bounded for the cut locus of any point on the manifold. This would now mean that the feedback term, which is essentially a geodesic gradient, when perturbed with the feed-forward term, (which is bounded due to isometry of parallel transport) would still make a strictly positive angle with ∂T_p, for any p. In general, this fact is far from obvious, because, even if the cut locus (or a segment) is smooth, it is possible that minimal geodesics tangentially meet it, and therefore any perturbation to the geodesic flow may result in intersecting the cut locus. Moreover, the structure of the cut-locus is itself quite irregular (may even be fractal), and therefore, characterizing the distance function to C_p, or its variation, is extremely difficult.

The neighborhood T_p considered above, is constructed by deforming a unit ball around p, via a *geodesic-homotopy*, which defines a *strong deformation retract* of the manifold punctured at p, on to the cut locus C_p. An advantage of doing this is that instead of considering the distance function to C_p, it suffices in considering the homotopy parameter s such that any point on a surface homotopic to the unit ball, lies outside C_p for $0 \leq s < 1$. The tubular neighborhood is essentially a level-set of s, and the angle bound is derived by showing that the gradient of s uniformly, strictly positively projects on to the geodesic tangent.

Finally, we show that the cut-locus C_p in a compact manifold, has a uniformly Hausdorff-stable variation with respect to p, and therefore for a bounded speed trajectory $p(t)$, $C_{p(t)}$ is uniformly Hausdorff stable in t. Hence, the feedback gain may be chosen large enough such that the homotopy parameter s decreases fast enough, with respect to the Hausdorff variation of C_p, and consequently the particle never intersects the cut-locus of $p(t)$, as long as it starts outside an arbitrarily small neighborhood, and therefore the closed loop system is almost-globally exponentially stable.