A landing theorem for transcendental functions with bounded postsingular set

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Conference
'On geometric complexity of Julia sets'
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Let f be an entire function, either polynomial or transcendental.

We will address the following question:

Question

What is the relation between the set of escaping points
$I(f) := \{z :\in \mathbb{C} : f^n(z) \to \infty\}$
and the set of repelling periodic points?

More precisely:

Question

Is it true that for every repelling periodic point z_0 there is a curve $\gamma \in I(f)$ which connects z_0 to infinity?

If f is a polynomial, $I(f)$ is the attracting basin of infinity, and if the critical points have bounded orbits, $I(f)$ is foliated by external rays: injective, disjoint curves $G_\theta(t) : \mathbb{R}_+ \to I(f)$ for which $|G_\theta(t)| \to \infty$ as $t \to \infty$.

An external ray G_θ is periodic if $f(G_\theta) = G_\theta$
An external ray G_θ lands at a point $z_0 \in \mathbb{C}$ if $G_\theta(t) \to 0$ as $t \to 0$

Douady-Hubbard Landing Theorem

Let f be a polynomial whose critical points have bounded orbits.

1. All periodic external rays land at repelling or parabolic periodic points
2. All repelling and parabolic periodic point are landing points of at least one (and at most finitely many) external rays, all of which are periodic.

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STANDING ASSUMPTION: $P(f)$ is bounded.

When f has good geometry and $S(f)$ is bounded, $I(f)$ consists of curves like in the polynomial case:

Theorem (Existence of rays: Rottenfusser, Rückert, Rempe, Schleicher 2010)

Let f with $S(f)$ bounded, f finite composition of functions of finite order. Then $I(f)$ consists of uncountably many curves $\{G_{\underline{s}}\}_{\underline{s}\in\mathbb{N}}$, called dynamic rays or hairs.

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In analogy with polynomial external rays, (dynamic) rays are unbounded, disjoint, injective curves in the set of escaping points,and they are equipped with symbolic dynamics.

- $f(G_{\underline{s}}) = G_{\sigma \underline{s}}$, where σ is the left-sided shift over $\mathbb{Z}^{\mathbb{N}}$.
- A ray $G_{\underline{s}}$ is *periodic* iff $f^n(G_{\underline{s}}) \subset G_{\underline{s}}$.
- A ray $G_{\underline{s}}$ *lands* iff $\overline{G_{\underline{s}}} \setminus G_{\underline{s}} = \{z_0\} \subset \mathbb{C}$.

Main Result

Landing Theorem for transcendental functions

Let f be a transcendental map for which rays exist, and such that $\mathcal{P}(f)$ is bounded.

1. All periodic rays land at repelling or parabolic periodic points; (Rempe 2008, Deniz 2014)
2. All repelling and parabolic periodic point are landing points of at least one (and at most finitely many) rays, all of which are periodic (B-Fagella 2016 for parabolic; B-Lyubich 2014 for exponentials).

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The assumption that $\mathcal{P}(f)$ is bounded is formally analogous to the assumption for Douady-Hubbard landing theorem, but it is in fact much stronger.

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The assumption that $\mathcal{P}(f)$ is bounded is formally analogous to the assumption for Douady-Hubbard landing theorem, but it is in fact much stronger.
The assumption that $P(f)$ is bounded is crucial. But what about the geometric assumption that ensure existence of rays?

Shrinking Lemma

When $\mathcal{P}(f)$ is bounded hence compact, $\mathbb{C} \setminus \mathcal{P}(f)$ is open and backward invariant, and f^{-1} contracts the hyperbolic metric on $\mathbb{C} \setminus \mathcal{P}(f)$: this gives us a very useful tool:

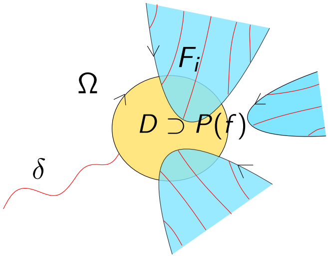
Shrinking Lemma (Spherical metric version)

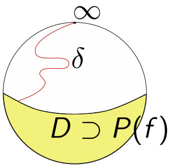
Let Ω be the unbounded connected component of $\mathbb{C} \setminus \mathcal{P}(f)$ and let $V \Subset Omega$ be a bounded Jordan domain. Then for any $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$, and all connected component U of $f^{-n}(V)$ we have

$$\operatorname{diam}_{\text{spher}} U < \varepsilon.$$

Dynamical plane for f

- $D \supset P(f)$
- $\Omega = \mathbb{C} \setminus D$
- Tracts T_i: connected components of $f^{-1}(\Omega)$, unbounded and simply connected
- $\delta \subset \Omega \setminus \bigcup T_i$
- $f : F_i \to \Omega \setminus \delta$ fundamental domains is univalent
- $\psi_F : \Omega \to F$ inverse
- Finitely many F_i intersect D





Construction of Rays

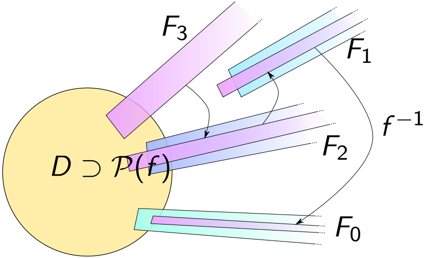
The family $\mathcal{F}$ of all fundamental domains induces a symbolic dynamics on the set of escaping points: fix $\underline{s} = F_0 F_1 F_2 \ldots$: at first guess, $G_{\underline{s}} := \{z : f^n(z) \in F_n\}$

Fix $\underline{s} = F_0 F_1 F_2 \ldots \in \{\mathcal{F}\}^{\mathbb{N}}$.
The fundamental tail of level n and address $\underline{s}$ is defined as:

$$\tau_n(\underline{s}) := f_{\underline{s}}^{-n+1}(F_{n-1})$$

The dynamic ray of address $\underline{s}$ is defined as:

$$G_{\underline{s}} \approx \{z \in I(f) : z \in \tau_n(\underline{s}) \text{ for } n \text{ large}\}.$$

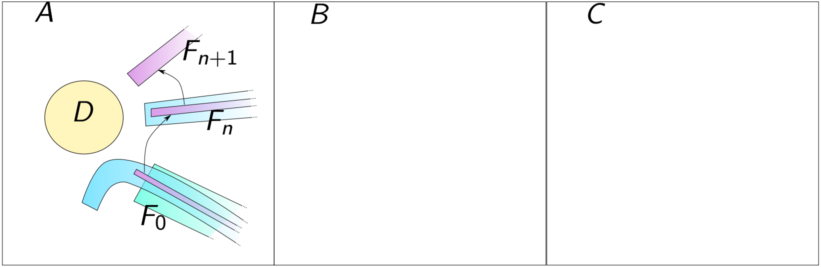


($f_{\underline{s}}$ identifies a specific set of inverse branches).

Remark

- $\tau_n(\underline{s})$ in fact only depends on the first n entries;
- $\tau_{n+1}(\underline{s}) \subset \tau_n(\underline{s}) \cup p_n$, where p_n is an extra piece which may be empty

Examples



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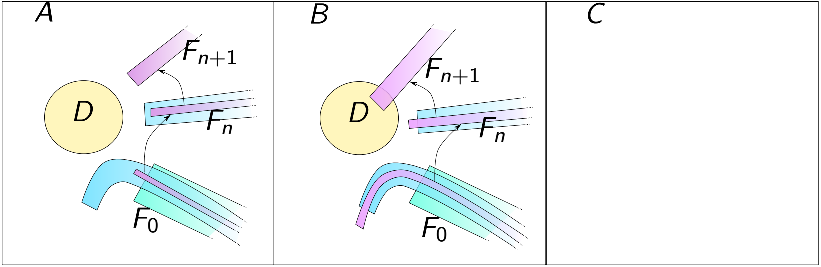

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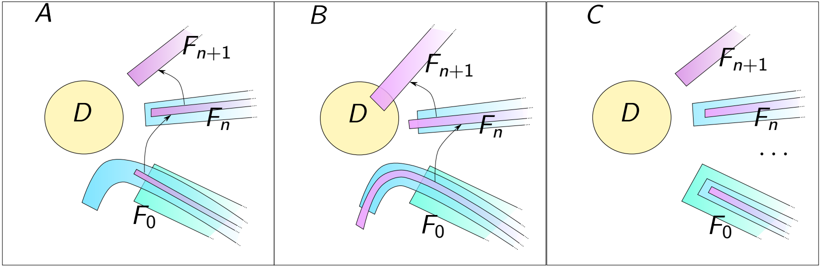

Characterization of landing

We say that $G_{\underline{s}}(t)$ lands at $z_0 \in \mathbb{C}$ if

- $G_{\underline{s}}(t) \to z_0$ as $t \to 0$.

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If $G_{\underline{s}}$ is periodic of period p, $G_{\underline{s}}$ lands at z_0 iff

- $f_{G_{\underline{s}}}^{-p}(\zeta) \to z_0$ for all $\zeta \in G_s$;

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by the shrinking lemma, this holds iff

- $f_{G_{\underline{s}}}^{-p}(\zeta) \to z_0$ for some $\zeta \in G_{\underline{s}}$.

A more abstract characterization of landing

Let $\underline{s}$ be an address, let $\zeta \in \Omega$ and let $\zeta_n(s)$ be the unique preimage of ζ which is contained in $\tau_n(\underline{s})$. Then, $G_{\underline{s}}$ lands at z_0 if and only if $\zeta_n(s) \to z_0$.

Lemma (Definition of the set V)

Let $\mathcal{F}$ be a finite collection of fundamental domains which includes all the ones which intersect D. Then there exists an open simply connected set $V \subset \mathbb{C} \setminus \mathcal{P}(F)$ which satisfies the following. Fix $\underline{s}$ and let $V_n(\underline{s})$ be the c.c. of $f^{-n}(V) \ni \zeta_n(\underline{s})$.

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- $G_{\underline{s}}$ lands at $z_0 \in \mathbb{C}$ iff $V_n(\underline{s}) \to z_0$;
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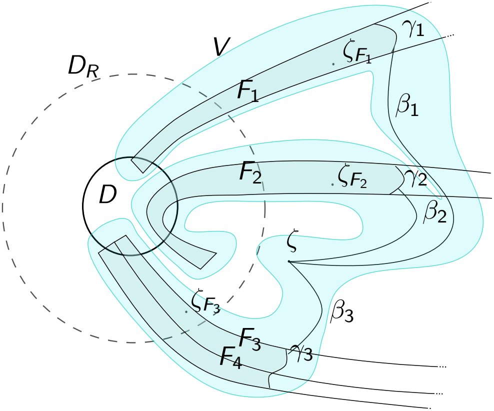
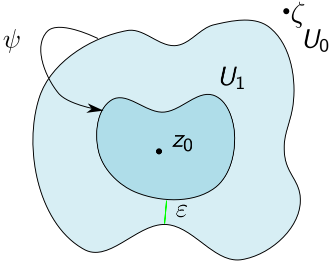


Instead of taking the exact branches which bring a periodic ray back to itself, for any ray we can take an analog set of branches which would approximate the correct ones in the periodic case. Instead of taking a point ζ on the ray, by the shrinking lemma I can take an arbitrary point in $\zeta \in \Omega$.

Setup of the main theorem

- z_0 repelling fixed point;
- $\zeta \in \Omega$ arbitrary;
- ψ inverse branch of f such that $\psi(z_0) = z_0$;
- U_0 a small neighborhood of z_0;
- $U_n := \psi^n(U_0) \Subset U_{n-1}$;
- $\varepsilon := \mathrm{dist}(\partial U_0, \partial U_1)$.



Strategy of the proof

- STEP 1 Find a pool of finite sequences which will be the building blocks of any candidate address.
- STEP 2 Show that the ray associated to any candidate address lands.

STEP 1 Find a candidate address

- Let V be the open connected set defined before, for the family of fundamental domains intersecting D.

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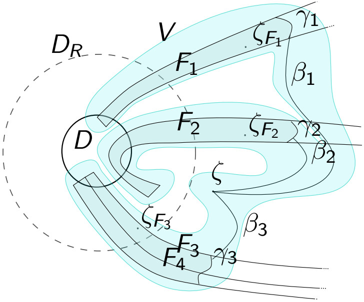

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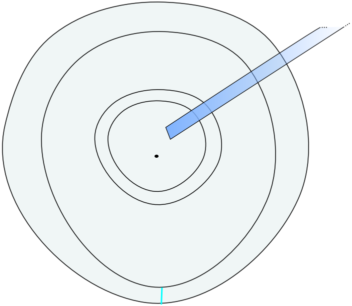
- For any $n \geq 0$ consider the tails of level $n+1$ which intersect U_n and let

$$\mathcal{S}_{n+1} := \{\text{finite sequences of length } n+1 \text{ for which } \tau_{n+1}(\underline{s}) \text{ intersect } U_n\}.$$

Claim

- For all $n \geq 1$, $\mathcal{S}_n \neq \emptyset$ and $\sigma : \mathcal{S}_{n+1} \to \mathcal{S}_n$ onto;
- If $F_0 F_1 \ldots F_n \in \mathcal{S}_{n+1}$ then $F_0 F_1 \ldots F_{n-1} \in \mathcal{S}_n$.

Proof by Picture



Proof of claim

Let $\tau_{n+1}(\underline{s}) \cap U_n \neq \emptyset$. Then

$$\tau_{n+1}(\underline{s}) \subset \tau_n(\underline{s}) \cup V_n(\underline{s}).$$

If $n = 1$, ok because $\mathrm{diam}(V_n(\underline{s})) < \varepsilon$. Otherwise map it forward $n-1$ times and then backwards to get $V_n(\underline{s}) \subset U_{n-1}$ hence $\tau_n(\underline{s}) \cap U_{n-1} \neq \emptyset$.

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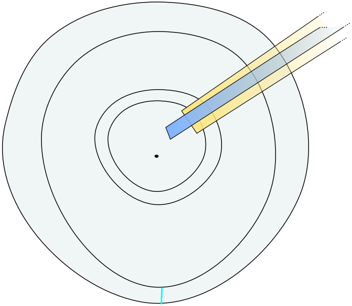

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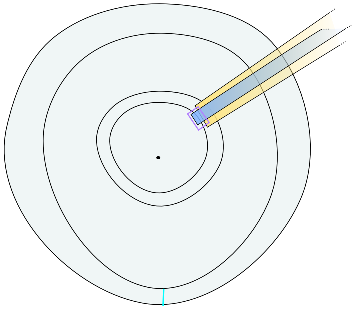

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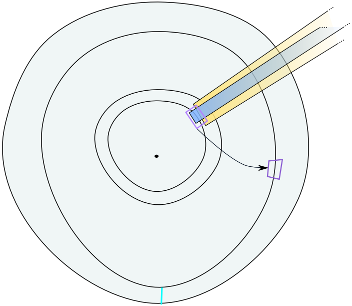

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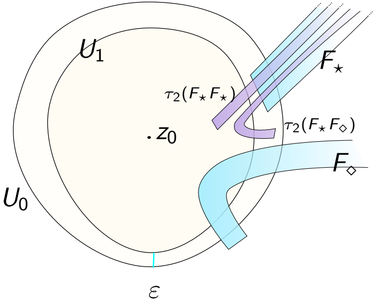
Example



Example 1

$$\mathcal{S}_1 = F_\star,\ F_\diamond$$
$$\mathcal{S}_2 = \boxed{F_\star F_\star},\ \boxed{F_\star F_\diamond},\ F_\diamond F_\star,\ F_\diamond F_\diamond$$
$$\mathcal{S}_3 = F_\star F_\star F_\star,\ F_\star F_\star F_\diamond$$
$$\mathcal{S}_4 = \ldots$$

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Second part of the proof

We obtain a graph over finitely many symbols

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in which we can find a limiting path corresponding to an infinite address $\underline{s} = \overline{F_\star F_\diamond}$.
Since

$$V_n(\underline{s}) \subset U_n \text{ for all } n$$

we have

$$V_n(\underline{s}) \to z_0$$

and $G_{\underline{s}}$ lands at z_0 by the abstract characterization of landing.

That is just the beginning...

That is just the beginning. . .

but let's make it the end!

THANK YOU!!!!