Complex Feigenbaum Phenomena of High Type

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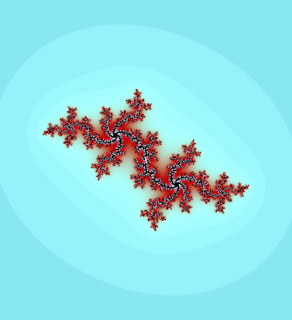
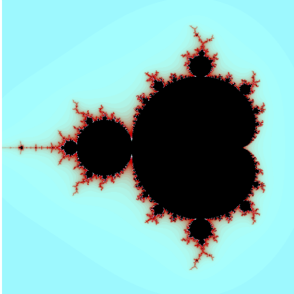
Basic Definitions

The main object of study is the dynamics of

$$P_c(z) = z^2 + c$$

Julia set $J(P_c) = \partial\{z \in \mathbb{C} \mid P_c^n(z) \text{ remains bounded}\}$
Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} \mid J(P_c) \text{ is connected}\}$





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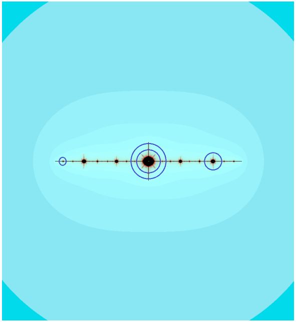
If P_c is at most finitely renormalizable with all periodic points repelling, then :

- $J(P_c)$ is locally connected. (Yoccoz)
- $J(P_c)$ has zero Lebesgue measure (Lyubich, Shishikura)

Polynomial-like Renormalization

P_c is PL-renormalizable if there is an integer $k > 1$ and simply connected domains $U \Subset V$ such that

- $P_c^k|_U : U \to V = P_c^k(U)$ is a proper branched covering map of degree two
- The little julia set, $\partial\{z \in U \mid P_c^{kn}(z) \text{ remains in } U\}$, is connected.



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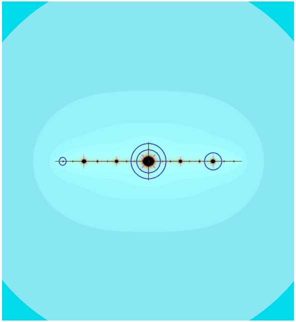
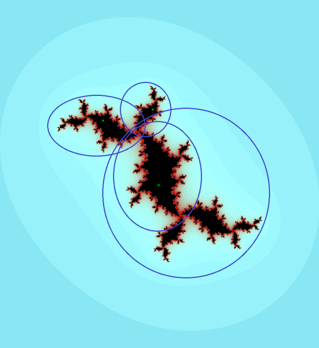
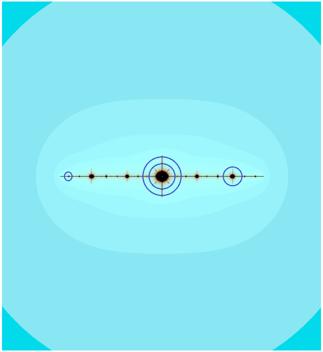
It follows from staightening theorem that $P_c^k|_U$ is topological (hybrid) conjugate to a (unique) quadratic $P_{c'}$.



Primitive and Satellite Renormalization

The little Julia sets for a renormalization are either pairwise disjoint or intersect at exactly one point. We refer to these as primitive and satellite types respectively.





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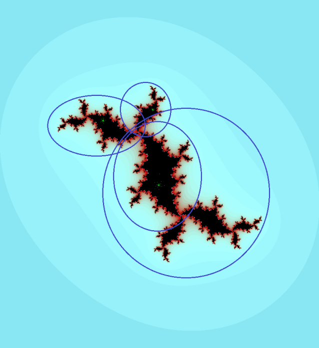

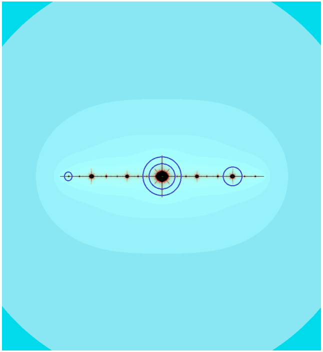

Each satellite renormalization can be combinatorially described by a rational number in $(-1/2, 1/2]$. Every such rational number is realized.

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Let $P_c^{k_n} : U_n \to V_n$ be the n-th PL-renormalization of P_c. Then P_c has a priori bounds if

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For real quadratics and several classes of infnitely renormalizable maps of primitive type, a priori bounds exists and J is locally connected (McMullen, Graczyk, Swiatek, Levin, van Strien, Kahn, Lyubich, Yampolsky, Jiang , et al ...)

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In 2009, Levin found an explicit condition on τ that guarantees non-local connectivity of the Julia set.

The Post Critical Set

One of the most useful sets to consider is

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Lyubich showed that for Lebesgue a.e. $z \in J(P_c)$ we have

$$\omega(z) \subset \mathcal{PC}(P_c)$$

High type combinatorics

Any rational number $\frac{p}{q} \in (-1/2, 1/2] \setminus \{0\}$ can be written as

$$\frac{p}{q} = \pm \cfrac{1}{b_1 \pm \cfrac{1}{\ddots \pm \cfrac{1}{b_n}}}$$

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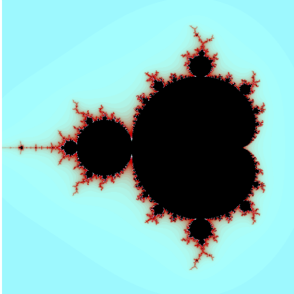
We define the high type combinatorics as :

$$\mathcal{HT}_N := \{\{p_i/q_i\}_{i=1}^{\infty} \mid b_{i,j} > N\}$$

For any sequence of rationals, τ, we can look at the sequence of roots of hyperbolic componenents, associated to τ.



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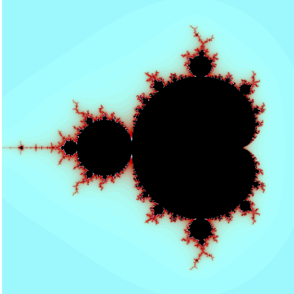

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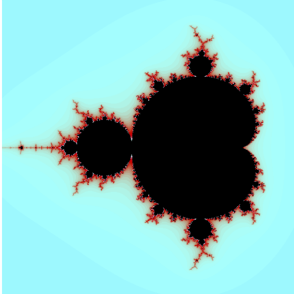

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The class of maps we consider is :

$$\mathcal{S}_N := \{P_{c(\tau)} \mid \tau \in \mathcal{HT}_N\}$$

A Dichotomy

Cheraghi, P. - 2017

There is $N \in \mathbb{N}$, such that for all $\tau \in \mathcal{HT}_N$, we have one of the following two statements:

- τ satisfies *generalised Herman-Yoccoz condition and* $\mathcal{PC}(P_{c(\tau)})$ *is a Cantor set of points,*
- τ *does not satisfy generalised Herman-Yoccoz condition and* $\mathcal{PC}(P_{c(\tau)})$ *is a hairy Cantor set.*

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Corollary

If $\tau \in \mathcal{HT}_N \setminus \mathcal{GHY}$, then $J(P_{c(\tau)})$ is not locally connected.

Herman-Yoccoz Condition

For $x \in \mathbb{R} \setminus \{0\}$, define

$$G(x) = d(\frac{1}{x}, \mathbb{Z})$$

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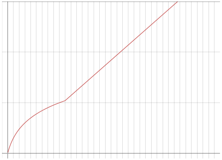
$$h_r(y) = \begin{cases} ry + \log(\frac{1}{r} + 1) - 1 & y \geq 1/r \\ \log(y + 1) & y < 1/r \end{cases}$$



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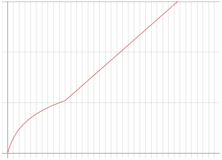

Then $\alpha \in \mathcal{H}$ iff

$$\lim_{n\to\infty} h_\alpha \circ h_{G(\alpha)} \circ \ldots \circ h_{G^{n-1}(\alpha)}(\mathcal{B}(G^n(\alpha))) = 0$$

Generalized Herman-Yoccoz Condition

For $r \in (0, 1/2]$ we define $g_r : \mathbb{R} \to \mathbb{R}$

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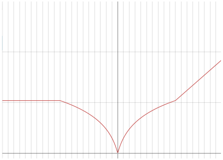

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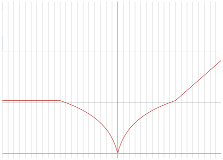

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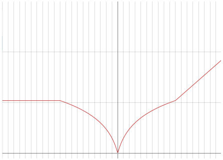

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$$\lim_{i \to \infty} (-g_{G(x_1)} \circ \ldots \circ g_{G^{n_1}(x_1)} \circ g_{|x_2|})) \circ \ldots \circ (-g_{G(x_i)} \circ \ldots \circ g_{G^{n_i}(x_i)} \circ g_{|x_{i+1}|}))(\frac{1}{|x_{i+1}|}) = 0$$

Hairy Cantor Set

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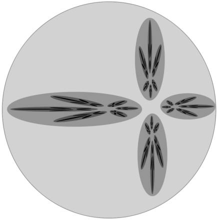

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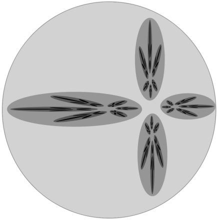

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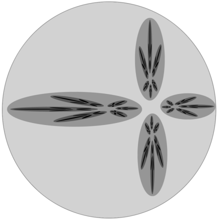

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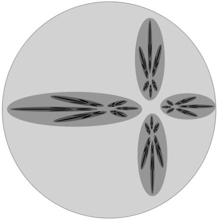

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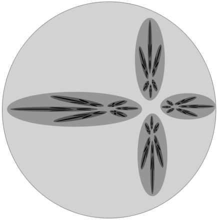

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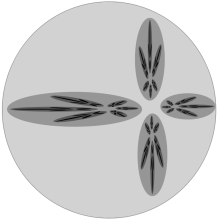

Cheraghi, P. - 2017

All hairy Cantor sets are homeomorphic.

Lack of a priori bounds

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For $\tau = \{p_i/q_i\}_{i=1}^{\infty} \in \mathcal{HT}_N$, let l_i denote the length of any geodesic of level i in $\mathbb{C} \setminus \mathcal{PC}(P_{\tau(c)})$, then

$$\left| l_i - \frac{1}{2\pi} \log(|\frac{q_i}{p_i}|) \right| < C$$

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Corollary

Let $\tau = \{p_i/q_i\}_{i=1}^{\infty} \in \mathcal{HT}_N$. Then $\mathcal{PC}(P_{c(\tau)})$ has *bounded geometry*, iff

$$|p_i/q_i| > D$$

for some constant $D > 0$ and all $i \in \mathbb{N}$.

Typical trajectories

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Let $f \in \mathcal{S}_N$. Then for Lebesgue almost every point $z \in J(f)$:

$$\omega(z) = \omega(cp_f) = \mathcal{PC}(f)$$

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For $f \in \mathcal{S}_N$, the post-critical set is non-uniformly porous. In particular

$$Area(\mathcal{PC}(f)) = 0$$

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- There are two operators $\mathcal{R}_t$ and $\mathcal{R}_b$ that act on an infinite dimensional class of maps.
- Renormalization towers corresponding to iterates of $\mathcal{R}_t$ can be used to study the dynamics of certain maps with an irrationally indifferent fixed point. This has been carried out successfully by Cheraghi in a series of papers.

Mixed renormalizations

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- There is more flexibility in the renormalization tower due to complex rotations near the two fixed points. Natural objects such as Fatou coordinates can not be lifted in the tower due to spiralling effect which occur due to complex rotations.

Arithmetic model for $\mathcal{PC}$

- We first build a model for $\mathcal{PC}(P_{c(\tau)})$, which does not use the iteration of the renormalization or the maps. It is a purely arithmetic model, which is essentially obtained by mimicking the behaviour of the changes of coordinates in the tower.

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- Then we build a homeomorphism between the model and the post-critical set. This is obtained from the limit of a Cauchy sequence of homeomorphisms. The advantage of this method is that we are able to study the regularity of the Jordan curves in the postcritial set.

Thank you !