Dynamics of Schwarz reflections: mating polynomials with groups

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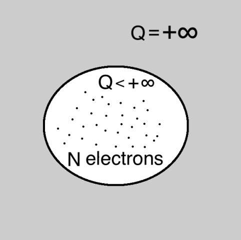

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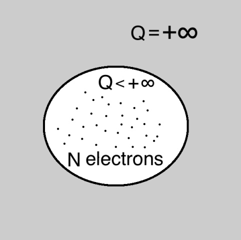

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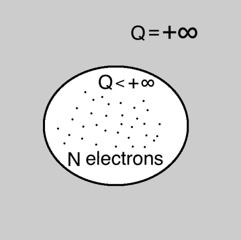

- The combined energy resulting from particle interaction and external potential is: $\mathcal{E}_Q(z_1, \cdots, z_N) = \sum_{i \neq j} \ln |z_i - z_j|^{-1} + N \sum_{j=1}^{N} Q(z_j)$.

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- In these situations, the complementary components of the droplet T admit global reflection maps. (Lee-Makarov)

From Algebraic Droplets to Quadrature Domains

Definition (Quadrature Domains)

A domain $\Omega \subsetneq \hat{\mathbb{C}}$ with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$ is called a *quadrature domain* if there exists a continuous function $\sigma : \overline{\Omega} \to \hat{\mathbb{C}}$ satisfying the following two properties:

1. $\sigma = \mathrm{id}$ on $\partial\Omega$.
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- Iteration of Schwarz reflections displays features of (anti-)rational maps as well as groups.

Simply Connected Quadrature Domains

Proposition

A simply connected domain $\Omega \subsetneq \hat{\mathbb{C}}$ *with* $\infty \notin \partial\Omega$ *and* $\operatorname{int}(\overline{\Omega}) = \Omega$ *is a quadrature domain if and only if the Riemann map* $\phi : \mathbb{D} \to \Omega$ *is rational.*

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- The rational map ϕ semi-conjugates the reflection map $1/\overline{z}$ of $\mathbb{D}$ to the Schwarz reflection map σ of Ω.

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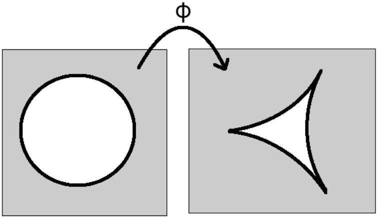
- The complement of the deltoid has a Riemann map $\phi(z) = z + \frac{1}{2z^2}$, so it is a quadrature domain.



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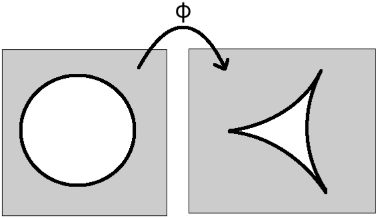
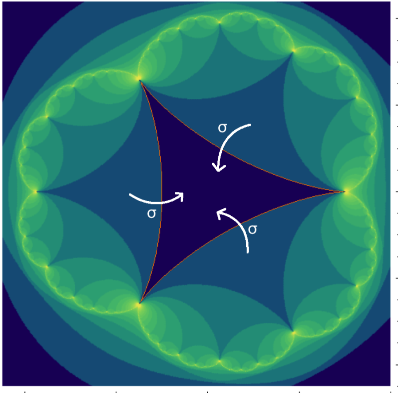


- The corresponding Schwarz reflection map is unicritical, and has a super-attracting fixed point at ∞.

Dynamics of Deltoid Reflection



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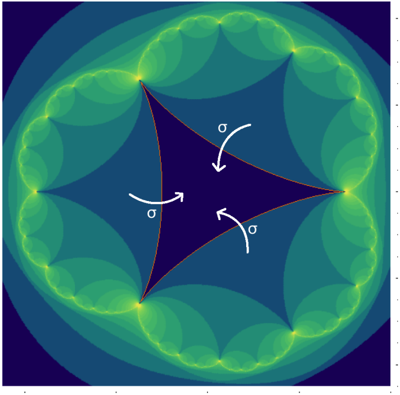

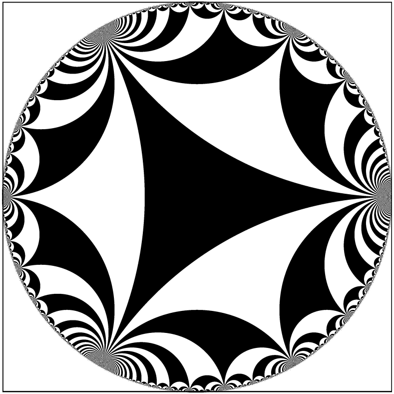

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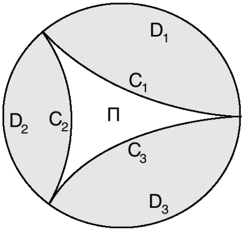

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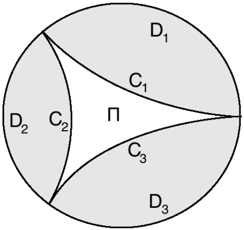

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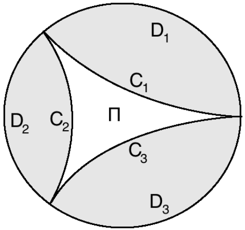

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- Define the reflection map $\rho : \mathbb{D} \setminus \Pi \to \mathbb{D}$ as:

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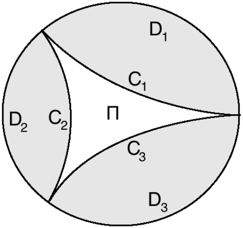

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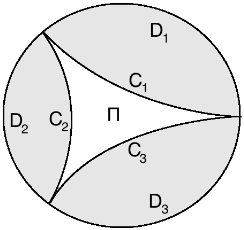

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- The dynamics of the deltoid reflection map is a "mating" of ρ (on the tiling set) and $\overline{z}^2$ (on the non-escaping set).

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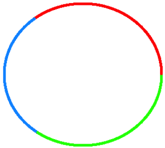

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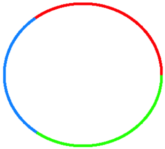

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$$M := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

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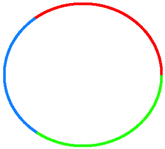

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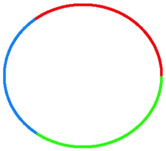

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- $\mathcal{H}$ "welds" the external class of quadratic antiholomorphic polynomials and that of the ideal triangle group.

The Circle and Cardioid Family

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- Let $\Omega_a := \heartsuit \cup \overline{B}(a, r_a)^c$. We now define our dynamical system $F_a : \overline{\Omega}_a \to \hat{\mathbb{C}}$ as,

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- We will call this family of maps $\mathcal{S}$; i.e.

$$\mathcal{S} := \{F_a : \overline{\Omega}_a \to \hat{\mathbb{C}} : a \in \mathbb{C} \setminus (-\infty, -1/12)\}.$$

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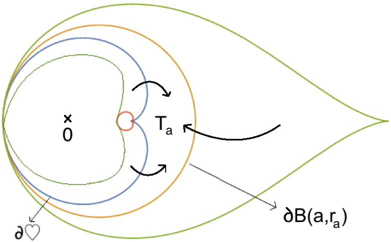

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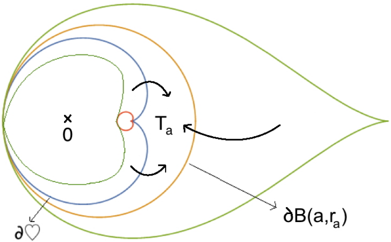

- The *tiling set* T_a^∞ of F_a is defined as the set of points in $\overline{\Omega}_a$ that eventually escape to T_a; i.e. $T_a^\infty = \displaystyle\bigcup_{k=0}^{\infty} F_a^{-k}(T_a^0)$.

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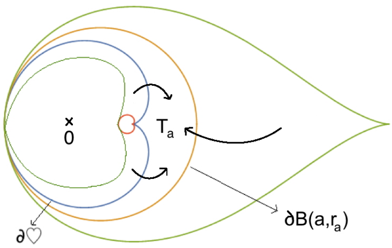

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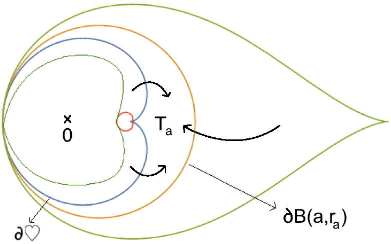

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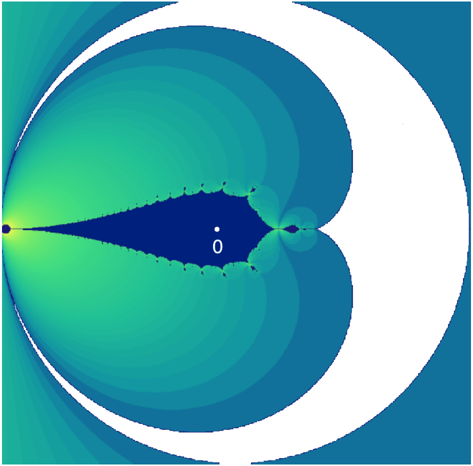
- The boundary of T_a^∞ is called the *limit set* of F_a, and is denoted by Γ_a.

Dynamical Plane of the Basilica Map



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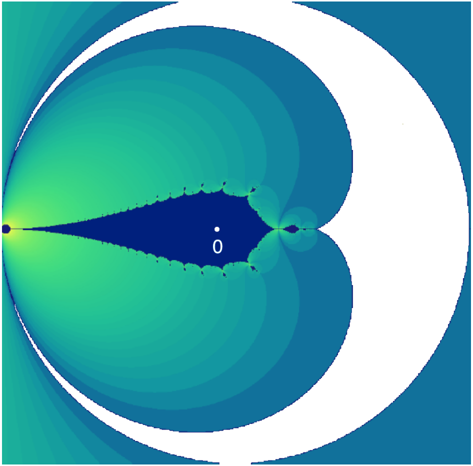
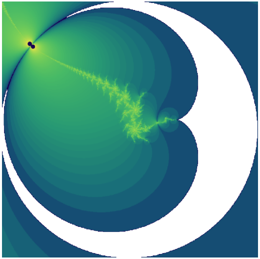


- $0 \mapsto \infty \mapsto 0$; the "Basilica" map.

Various Dynamical Planes



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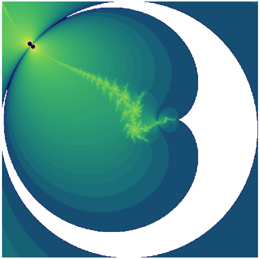

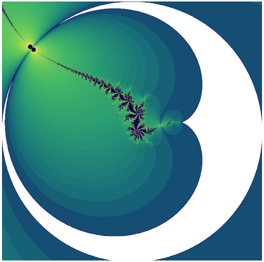

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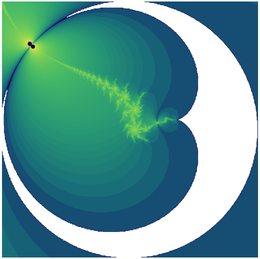

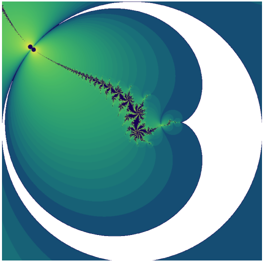

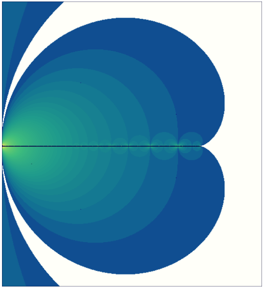

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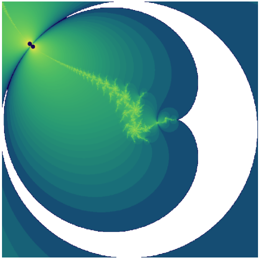

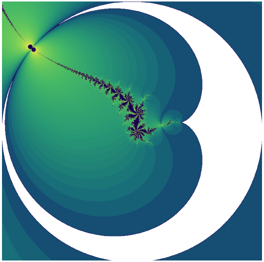

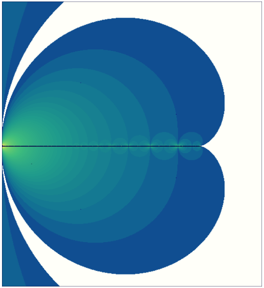

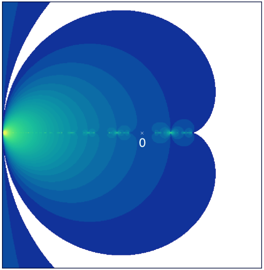

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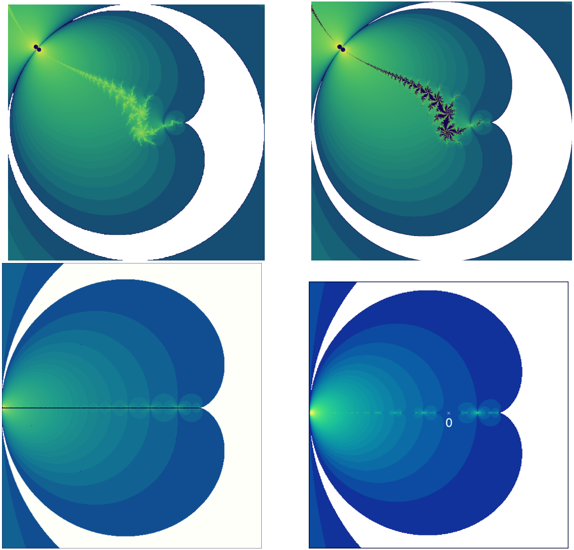

- "Full tessellation structure" iff tiles are unramified; i.e. $0 \notin T_a^\infty$.

The Connectedness Locus $\mathcal{C}(\mathcal{S})$

- $\mathcal{C}(\mathcal{S}) = \{a \in \mathbb{C} \setminus (-\infty, -1/12) : K_a \text{ is connected}\}$
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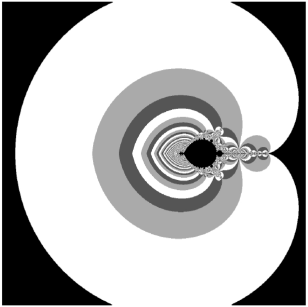

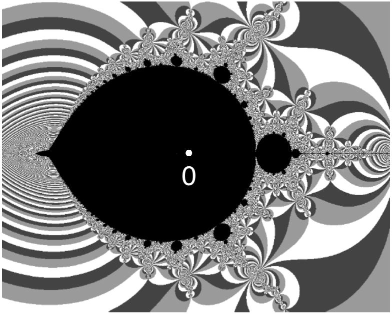

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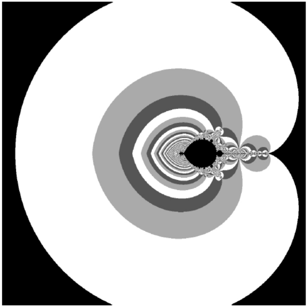

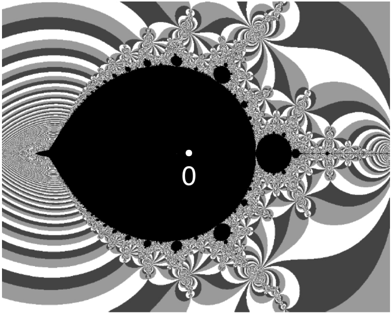

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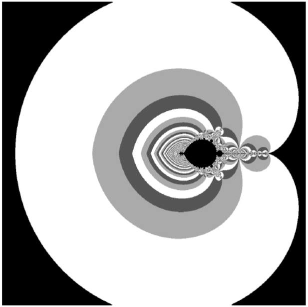

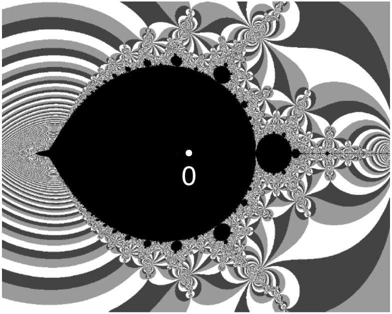

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- For maps outside $\mathcal{C}(\mathcal{S})$, the "Böttcher coordinate" is defined on low generation tiles.

External Dynamics and Lamination Model of Limit Sets

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- This yields a pinched disk model of the non-escaping set K_a via a ρ-invariant geodesic lamination of $\mathbb{D}$.
- Rigidity: PCF maps are uniquely determined by laminations. The proof is based on the "Pullback Argument", and involves an analysis of the boundary behavior of conformal maps near cusps and double points.

Bijection between PCF Parameters

Theorem

There exists a natural combinatorial bijection between the PCF parameters of $\mathcal{S}$ and PCF anti-quadratic polynomials such that the laminations of the corresponding maps are related by the circle homeomorphism $\mathcal{H}$.

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- Surjectivity: Finding PCF maps in $\mathcal{S}$ with prescribed laminations. We achieve this "from outside"; i.e. as limit points of suitable sequences of parameter tiles.

PCF Maps are Mating

Theorem

Every PCF map F_a is a mating of the PCF quadratic anti-holomorphic polynomial $f_{\chi(a)}$ and the reflection map ρ. More precisely, F_a is topologically conjugate to $f_{\chi(a)}$ on its non-escaping set, and conformally conjugate to ρ on its tiling set.

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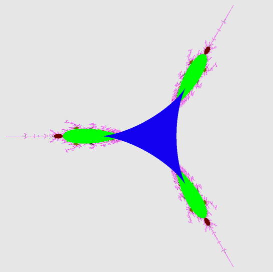

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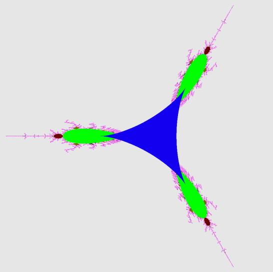

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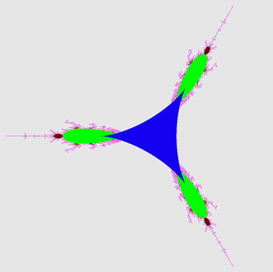

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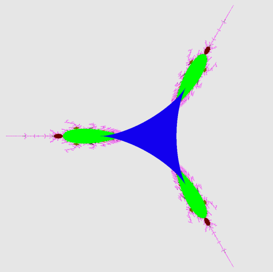

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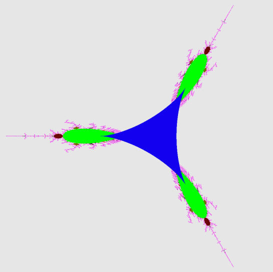

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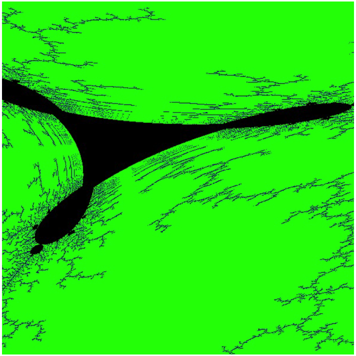
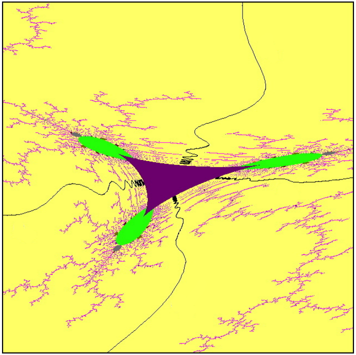
- Many rational parameter rays non-trivially accumulate. Misiurewicz parameters are not dense on the boundary. (Inou-M)
- "Baby tricorns" are not homeomorphic to the original tricorn. (Inou-M)

Topology of the Tricorn





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- Landing/accumulation patterns of parameter rays precisely correspond under $\mathcal{H}$.
- This descends to a homeomorphism between the pinched disk models of connectedness loci.

Thank you!