Algebraic Geometry - Mariusz Koras in memoriam
Warsaw, 28.05.2018 - 01.06.2018

ABSTRACTS

Maciej Borodzik, *University of Warsaw*

Heegaard Floer homologies and rational cuspidal curves

ABSTRACT: We find restrictions on semigroups of singular points of rational cuspidal curves in $\mathbb{CP}^2$, related to the conjecture of Fernandez de Bobadilla, Luengo, Melle-Hernandez and Nemethi. The methods that are used are essentially topological. Generalizations apply for non-rational case. This is a joint work with Chuck Livingston and Matt Hedden

Pierrette Cassou–Noguès, *IMB, Universite de Bordeaux*

Polynomials in two variables and their tree at infinity

ABSTRACT: Let $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a polynomial map. Let $\mathbb{C}^2 \subset X$ be a compactification of $\mathbb{C}^2$, where X is a smooth rational compact surface such that there exists a holomorphic map $\phi\colon X \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ which extends f. Put $\mathcal{D} = X \setminus \mathbb{C}^2$. $\mathcal{D}$ is a curve whose irreducible components are smooth rational compact curves and all singularities are ordinary double points. The dual graph of $\mathcal{D}$ is a tree. In this talk we will show that the genus of the generic curve is an increasing function of the complexity of the tree using combinatorial methods.

This is a joint work with Daniel Daigle.

Adrien Dubouloz, *CNRS, Institut de Mathématiques de Bourgogne*

Koras-Russell threefolds: Geometry, Algebra, Topologies and beyond

ABSTRACT: These threefolds have played a central role in the development of higher dimensional affine algebraic geometry during the last decades. They more recently attracted the attention of other fields in geometry, such as symplectic topology and $\mathbb{A}^1$-homotopy theory. In this panoramic talk, I will review many classical as well as much less known properties of these threefolds, with a particular focus on potential interactions with other fields of research and open problems.

Gene Freudenburg, *Western Michigan University*

The polar group of a real form of an affine $\mathbb{C}$-variety

ABSTRACT: Over several years, Koras together with Cassou-Nogues, Palka and Russell worked to classify embeddings of $\mathbb{C}^*$ in $\mathbb{C}^2$. This classification was completed only recently. Using their classification, the author shows that any polynomial embedding of the real 1-sphere $\mathbb{S}^1$ in $\mathbb{R}^2$ is equivalent to the standard embedding. Polar groups of real forms are introduced for this proof. This group is a natural invariant of a real form of an affine $\mathbb{C}$-variety, and it will be discussed in this talk. In addition, we conjecture that (1) all polynomial embeddings of $\mathbb{S}^1$ in $\mathbb{R}^3$ are equivalent, and (2) the torus $\mathbb{S}^1 \times \mathbb{S}^1$ does not admit a polynomial embedding in $\mathbb{R}^3$.

Rajendra Gurjar, *Indian Institute of Technology*

A new proof of C.P. Ramanujam's characterization of the affine 2-space

Abstract: We will give a new proof of C.P. Ramanujam's topological characterization of the affine 2-space using some standard results from the theory of non-complete algebraic surfaces.

This is a joint work with my son Sudarshan Gurjar.

Isac Hedén, *University of Warwick*

Extensions of principal additive bundles over a punctured surface

Abstract: We study complex affine $\mathbb{G}_a$-threefolds X such that the restriction of the quotient morphism $\pi\colon X \longrightarrow S$ to $\pi^{-1}(S_*)$ is a principal $\mathbb{G}_a$-bundle, where $S_* = S \setminus \{o\}$ denotes the complement of a closed point $o \in S$ and $\mathbb{G}_a$ denotes the additive group over the field of complex numbers. Changing the point of view, we look for affine extensions of $\mathbb{G}_a$-principal bundles over punctured surfaces, i.e. affine varieties that are obtained by adding an extra fiber to the bundle projection over o. Special attention will be given to the case where X is smooth, the $\mathbb{G}_a$-action on X is proper and $\pi^{-1}(o) = \mathbb{A}^2$ is the affine plane.

Zbigniew Jelonek, *Polish Academy of Sciences*

Simple examples of affine manifolds with infinitely many exotic models

Abstract: We give a simple general method of constructing affine varieties with infinitely many exotic models. In particular we show that for every $d > 1$ there exists a Stein manifold of dimension d which has uncountably many different structures of affine variety.

Shulim Kaliman, *University of Miami*

Complete algebraic vector fields on affine surfaces

Abstract: Let $\mathrm{AAut}_{\mathrm{hol}}(X)$ be the subgroup of the group $\mathrm{Aut}_{\mathrm{hol}}(X)$ of holomorphic automorphisms of a normal affine algebraic surface X generated by elements of flows associated with complete algebraic vector fields. Our main result is a classification of all normal affine algebraic surfaces X quasi-homogeneous under $\mathrm{AAut}_{\mathrm{hol}}(X)$ in terms of the dual graphs of the boundaries $\bar{X} \setminus X$ of their SNC-completions $\bar{X}$.

Takashi Kishimoto, *Saitama University*

Forms of the quintic del Pezzo threefold V_5

Abstract: Let k be a field of characteristic zero. In this talk, we look into a k-form X of the quintic del Pezzo threefold $X_{\bar{k}} = X_k \otimes \bar{k} = V_5$. One of main results asserts that any k-form X of V_5 contains an $\mathbb{A}^2_k$-cylinder. Further, we obtain a certain criterion for X to contain $\mathbb{A}^3_k$ in terms of the Hilbert scheme of lines on V_5 with respect to the half-anti canonical embedding. As an application, we see that every Mori Fiber Space over an algebraic complex curve whose general closed fibers are isomorphic to V_5 contains a *vertical* $\mathbb{A}^3_{\mathbb{C}}$-cylinder. This is a joint work with Adrien Dubouloz.

Hideo Kojima, *Niigata University*

Normal log canonical del Pezzo surfaces of rank one

Abstract: De-Qi Zhang initiated the systematic study of log del Pezzo surfaces of rank one in 1980s. It has been used to study many properties of such surfaces. In 2009, I and Takeshi Takahashi proved that Zhang's fundamental results on log del Pezzo surfaces of rank one hold true for normal del Pezzo surfaces of rank one with rational singular points. In this talk, I will report our recent progress on normal del Pezzo surfaces of rank one with rational singular points. In particular, I will give some partial classification results of normal log canonical del Pezzo surfaces of rank one by using results of Karol Palka on $\mathbb{Q}$-homology planes.

Frank Kutzschebauch, *University of Bern*

The Koras-Russell cubic threefold from the holomorphic point of view

Abstract: We will explain the density property for a Stein manifold X, introduced by Varolin, This property is a precise way of saying that the holomorphic automorphism group of X is large. The first and guiding example for such a manifold is $\mathbb{C}^n$ for $n \geq 2$. A conjecture by Varolin and Toth states that a Stein manifold with density property which is diffeomorphic to $\mathbb{R}^{2n}$ has to be biholomorphic to $\mathbb{C}^n$.

Next we present techniques to prove the density property which were developed by Kaliman and the speaker. Finally we show how to apply those techniques to prove the density property for the Koras-Russell cubic threefold, thus producing a potential counterexample to the Varolin-Toth-Conjecture.

This is a result of our former student Leuenberger.

Leonid Makar-Limanov, *Wayne State University*

An estimate of the geometric degree of a two-dimensional Jacobian mapping

Abstract: I'll explain how research of the Newton polytope of the irreducible relation of x, f, g, where f, g is a Jacobian pair, allows one to get an estimate of the geometric degree of the mapping $(x, y) \longrightarrow (f, g)$. I started this project when Mariusz and I were at the Fireflies' meeting in India in 2008 after a discussion with him.

Masayoshi Miyanishi, *Kwansei Gakuin University*

Affine space fibrations on algebraic varieties

Abstract: Let n be a positive integer and let X be an algebraic variety. An $\mathbb{A}^n$-fibration on X is a dominant morphism $f : X \longrightarrow Y$ such that f induces a regular extension $k(Y) \hookrightarrow k(X)$ of the function fields and general fibers of f are isomorphic to the affine n-space. It is the Dolgachev-Weisfeiler problem to ask if the generic fiber $X_\eta := X \times_Y \operatorname{Spec} k(Y)$ is isomorphic to $\mathbb{A}^n$ over the field $k(Y)$. If $n = 1, 2$ the answer is positive, but we do not know answers if $n \geq 3$. If $n = 1$ and $\dim X = 2$, the existence of $\mathbb{A}^1$-fibrations have played significant roles to clarify the structure of affine algebraic surfaces in affine algebraic geometry. Further, affine space fibrations are closely related to the quotient morphisms of algebraic unipotent group actions on X.

In the talk, we consider $\mathbb{A}^1$-fibrations ($\mathbb{A}^2$-fibrations as well) on affine threefolds with emphasis placed on the relations with $\mathbb{G}_a$-actions and singular fibers (characterizations of irreducible components, structure of two-dimensional components, locus of singular fibers on Y, etc.). Of particular interest is the existence of $\mathbb{A}^1$-fibrations in the complements of ample effective divisors in Fano threefolds.

Lucy Moser–Jauslin, *Université de Bourgogne*

A survey on automorphism groups of Koras-Russell threefolds

Abstract: Koras-Russell threefolds are smooth contractible complex affine $\mathbb{C}^*$-varieties of dimension three. Koras and Russell introduced these varieties 20 years ago in order to prove their remarkable result: every algebraic action of $\mathbb{C}^*$ on affine complex three-space is linearizable. Since then, the study of Koras-Russell threefolds has led to many important insights and results in the theory of affine algebraic geometry.

In this talk, I will give an overview of results concerning automorphism groups of a Koras-Russell threefolds and some of their consequences.

Sabrina Pauli, *University of Oslo*

$\mathbb{A}^1$-contractibility of Koras-Russell like varieties

Abstract: Koras-Russell threefolds first appeared in Koras and Russell's monumental work of showing that every $\mathbb{G}_m$-action on $\mathbb{C}^3$ is linearizable [KR97]. They are smooth, complex, topologically contractible, affine varieties of dimension three not isomorphic to $\mathbb{C}^3$ [ML96] which makes them potential counterexamples to the Zariski cancellation problem [Kra96]. In [HKØ16] it is shown that Koras-Russell threefolds of the first and second kind are stably $\mathbb{A}^1$-contractible and consequently $\mathbb{A}^1$-contractible after a finite suspension by the projective line $\mathbb{P}^1$- pointed at infinity. They are the first examples of smooth, complex, affine varieties not isomorphic to affine space which are stably $\mathbb{A}^1$-contractible. Koras-Russell threefolds of the first kind are in fact $\mathbb{A}^1$-contractible as shown in [DF15]. We extend the results in [HKØ16] and [DF15] to a bigger family of hypersurfaces in $\mathbb{C}^4$ including Koras-Russell threefolds of the first kind, threefolds with several degenerate fibers as well as hypersurfaces given by

$$x^n z = y^{\alpha_1} + t^{\alpha_2} + xq(x, t, y), \tag{1}$$

where $n, \alpha_1, \alpha_2 \geq 2$ and $\gcd(\alpha_1, \alpha_2) = 1$, and $q(x, y, t) \in \mathbb{C}[x, y, t]$ with $q(0, 0, 0) \neq 0$. The techniques in [HKØ16] and [DF15] can also be applied to hypersurfaces in $\mathbb{C}^4$ given by

$$x^n z = (x^m y + z^s)^{\alpha_1} + t^{\alpha_2} + x \tag{2}$$

with $n, m, s, \alpha_1, \alpha_2 \geq 2$ and $\gcd(s\alpha_1, \alpha_2) = 1$ obtained by an affine modification [KZ99] of a Koras-Russell threefold of the first kind. Additionally, we provide results on a four dimensional example, namely the hypersurface in $\mathbb{C}^5$ given by

$$x^n z = u^2 + t^3 + w^5 + x \tag{3}$$

with $n \geq 2$. The given examples (1), (2) and (3) are smooth, topologically contractible, complex, affine varieties not isomorphic to affine space which turn out to be stably $\mathbb{A}^1$-contractible, making them potential counterexamples to the Zariski cancellation problem.

References

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Karol Palka, *Polish Academy of Sciences*

On the number of cusps of rational cuspidal curves

Abstract: Let $E \subseteq \mathbb{P}^2$ be a complex algebraic curve homeomorphic to a line. Many examples with 1, 2 or 3 cusps are known, often belonging to rather exotic series. But assume that E has at least four cusps. It is conjectured that then the only possibility is that E is a (projectively unique) quintic.

Let (X, D) denote the minimal log resolution of the above curve. We will sketch the proof that the pair $(X, \frac{1}{2}D)$ is almost minimal and we will show how this fact leads to a finite list of possible singularity types of E. The proof of the conjecture should follow soon. This is a continuation of a joint project with Mariusz Koras.

Tomasz Pełka, *Polish Academy of Sciences*

Classifying smooth $\mathbb{Q}$-homology planes

Abstract: A $\mathbb{Q}$-homology plane ($\mathbb{Q}$HP) is a smooth affine surface S such that $H_i(S, \mathbb{Q}) = 0$ for $i > 0$. Such surfaces are well understood if $\kappa(S) < 2$, however, no structure theorems are known if $\kappa(S) = 2$. One of the main difficulties comes from the fact that in this case any smooth completion (X, D) of S is almost minimal, so the application of log MMP does not give much insight to its structure. To overcome this obstacle, K. Palka proposed to study the pair $(X, \frac{1}{2}D)$ instead. This approach leads to the Negativity Conjecture, which asserts that $\kappa(K_X + \frac{1}{2}D) = -\infty$, so a minimal model of $(X, \frac{1}{2}D)$ is a log Mori fiber space. This conjecture generalizes the Rigidity Conjecture of Flenner and Zaidenberg.

I will explain how to use log MMP for $(X, \frac{1}{2}D)$ to study of $\mathbb{Q}$HPs satisfying this conjecture. In particular, I will show that every such surface admits a $\mathbb{C}^{t*}$-fibration for some $t \leq 3$. I will also sketch the full classification in case when $S = \mathbb{P}^2 \setminus \bar{E}$ is a complement of a planar cuspidal curve. Eventually, I will indicate other directions where the log MMP with half-integral coefficients can be applied.

This is a joint work with Karol Palka.

Alexander Perepechko, *Kharkevich Institute for Information Transportation Problem*

Nested automorphism groups

Abstract: This talk is based on joint works with Sergey Kovalenko, Andriy Regeta, and Mikhail Zaidenberg. We will discuss complexity of automorphism groups of affine algebraic varieties and propose the following distinctive feature: whether unipotent elements generate an abelian subgroup. More precisely, we conjecture that the following conditions on the neutral component of the automorphism group are equivalent:

(1) it is equal to the union of all algebraic subgroups;
(2) it is exhausted by an inductive limit of algebraic subgroups (then we call it *nested*);
(3) it is a semidirect product of an algebraic torus and an abelian unipotent group;
(4) its tangent algebra consists of locally finite elements;
(5) its unipotent elements comprise an abelian subgroup.

We have fully confirmed this conjecture in dimension 2 and partially in arbitrary dimension: the equivalences hold for a subgroup of the automorphism group generated by algebraic groups (a subalgebra of the tangent algebra generated by locally finite elements, respectively)

Pierre–Marie Poloni, *Universität Bern*

On some affine plane bundles over the punctured affine plane

Abstract: An $\mathbb{A}^2$-fibration is a flat morphism between complex algebraic varieties whose fibers are isomorphic to the complex affine plane $\mathbb{A}^2$. In this talk, we study explicit families $f\colon \mathbb{A}^4 \longrightarrow \mathbb{A}^2$ of $\mathbb{A}^2$-fibrations over $\mathbb{A}^2$.

The famous Dolgachev-Weisfeiler conjecture predicts that such fibrations are in fact all isomorphic to the trivial bundle over $\mathbb{A}^2$. Our aim is to develop tools for verifying that this conjecture holds true in some particular examples. For instance, we will recover a result of Drew Lewis which states that the $\mathbb{A}^2$-fibration induced by the second Vénéreau polynomial is trivial.

Our strategy is inspired by a previous work of Kaliman and Zaidenberg and consists in first showing that the considered fibrations $f\colon \mathbb{A}^4 \longrightarrow \mathbb{A}^2$ have a fiber bundle structure when restricted over the punctured plane $\mathbb{A}^2 \setminus \{(0,0)\}$.

This is joint work in progress with Jérémy Blanc.

Vladimir Popov, *Steklov Mathematical Institute of the Russian Academy of Sciences*

Cremona groups vs. algebraic groups

Abstract: Although the Cremona groups are infnite-dimensional, the analogies between them and affine algebraic groups catch the eye: they have the Zariski topology, algebraic subgroups, tori, roots, Weyl groups, Borel subgroups, . . . The talk is aimed to discuss the similarities and differences related to this phenomenon.

Andryi Regeta, *University of Cologne*

Automorphism groups of Danielewski surfaces

Abstract: In this talk we will discuss the automorphism groups of Danielewski surfaces. It is known that two smooth generic Danielewski surfaces D_p and D_q have isomorphic automorphism groups. Moreover, the natural isomorphism $F\colon \operatorname{Aut}(D_p) \longrightarrow \operatorname{Aut}(D_q)$ restricted to any algebraic subgroup $G \subset \operatorname{Aut}(D_p)$ is an isomorphism of algebraic groups G and $F(G)$. On the other hand, in the joint work with Matthias Leuenberger we prove that groups $\operatorname{Aut}(D_p)$ and $\operatorname{Aut}(D_q)$ are isomorphic as ind-groups if and only if varieties D_p and D_q are isomorphic.

Furthermore, we will present the following result: let X be an affine normal variety such that $\operatorname{Aut}(X)$ is isomorphic to $\operatorname{Aut}(D_p)$ as an ind-group, then X is isomorphic to D_p as a variety.

Peter Russell, *McGill university*

$\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable
a somewhat historical overview of the proof and its ramifications

Abstract: I want to talk about the following, seemingly modest, result:

Let k be a field of characteristic zero. Then up to a choice of variables (x, y, z), a grading of the 3-dimensional polynomial algebra $k^{[3]}$ is defined by assigning integer weight (a, b, c) to the variables.

In more high-brow words, an algebraic action of the multiplicative group $G = \mathbb{G}_m$ of k on affine 3-space $X = \mathbb{A}^3$ is given by $t \cdot (x, y, z) = (t^a x, t^b y, t^c z)$ in suitable coordinates. This result has a surprising number of parents, Mariusz Koras prominent among them. It draws on a large number of results in algebraic geometry. For instance, already the fact that the fixed point set of the action is non-empty is non-trivial. Also, several new methods developed for the proof have inspired very nice new lines of investigation in related areas.

Let me add that the corresponding problem in dimensions 1 and 2 is much easier to resolve, but is open in dimensions 4 or higher, or in positive characteristic.

I will try to tell this story from a somewhat historical perspective, without becoming overly technical.

Christian Urech, *Imperial College London*

Characterization of toric surfaces by their automorphism groups

Abstract: In this joint work with Andriy Regeta and Alvaro Liendo we show that affine toric surfaces are determined by their automorphism group, i.e. if S_1 is an affine toric surface and S_2 a normal affine surface such that the groups $\mathrm{Aut}(S_1)$ and $\mathrm{Aut}(S_2)$ are isomorphic, then S_1 and S_2 are isomorphic. I will explain how to prove this kind of results using group theoretical properties of Cremona groups.

Mikhail Zaidenberg, *Institut Fourier, Grenoble*

Zariski cancellation problem for surfaces

Abstract: The Zariski Cancellation Problem asks when a stable isomorphism of affine varieties over an algebraically closed field implies an isomorphism. This is true for affine curves (Abhyankar, Eakin, and Heinzer '72), for the affine plane in zero characteristic (Miyanishi-Sugie and Fujita '79-'80), but false for general affine surfaces in zero characteristic (Danielewski '88) and for the affine space $\mathbb{A}^3$ in positive characteristic (N. Gupta '13). The talk is devoted to a recent progress in the surface case over a field of zero characteristic (Bandman-Makar-Limanov, Dubouloz, Flenner and Kaliman, et.al). It occurs to be possible to describe the moduli space of pairs of surfaces with isomorphic cylinders.