Polynomials in two variables and their trees at infinity

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Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial map.

Let $\mathbb{C}^2 \subset X$ be a compactification of $\mathbb{C}^2$ where X is a smooth rational compact surface and such that there exists a holomorphic map $\Phi : X \to \mathbb{P}^1$ which extends f. Put $\mathcal{D} = X \setminus \mathbb{C}^2$; $\mathcal{D}$ is a curve whose irreducible components are smooth rational compact curves and all its singularities are ordinary double points. The dual graph is a tree. We are interested in this tree, and we want to analyse its shape.

THE TREE

1: Resolution of singularities

Let $f(x,y) = a_0(x,y) + a_1(x,y) + \cdots a_d(x,y)$ where $a_j(x,y)$ is homogeneous of degree j. Let $F(x,y,z) = \sum_0^d z^{d-j} a_j(x,y)$. Consider the rational map:

$$\begin{array}{rcccc} \Phi_0 & : & \mathbb{P}^2 & \to & \mathbb{P}^1 \\ & & [x:y:z] & \mapsto & [F(x,y,z):z^d] \end{array}$$

It is well defined outside

$$A(f) = \{[x_0 : y_0 : 0] | a_d(x_0, y_0) = 0\}$$

The composition $\pi : X \to \mathbb{P}^2$ of a suitable sequence of blowing-up maps over $A(f)$ gives the required compactification, where $\Phi = \Phi_0 \circ \pi$.

$$\mathcal{D} = \pi^{-1}(\mathbb{L}_\infty)$$

Let E be an irreducible component of $\mathcal{D}$. If $\Phi(E) = \mathbb{P}^1$, we say that E is a dicritical component of $\mathcal{D}$. We denote by $\mathcal{D}_{dic}$ the set of dicritical components.

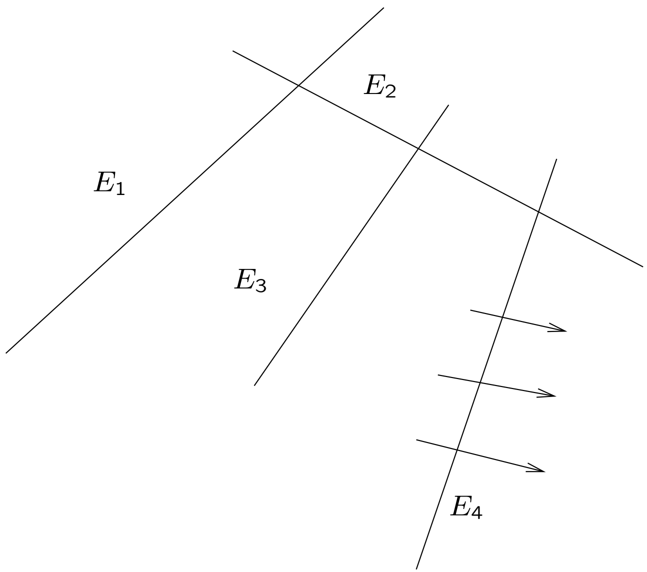
We set $\mathcal{D}_\infty = \Phi^{-1}(\infty)$. We have that $\mathcal{D}_\infty$ is connected and every component of $\mathcal{D}_{dic}$ intersects $\mathcal{D}_\infty$.

If $\Phi(E) = \infty$, we denote by $m(E)$ the order of the pole of Φ on E. If E is a dicritical we set $m(E) = 0$.

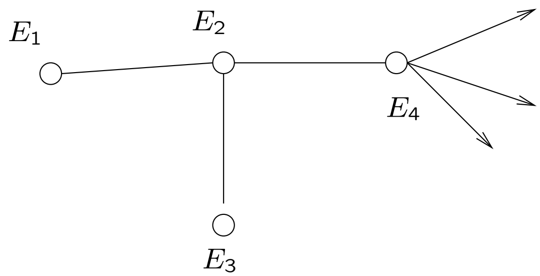
The affine curve $f(x, y) = t$, t generic intersects each dicritical in a finite number of points that we call the degree of the dicritical.

2: The tree.

We consider the dual graph of $\mathcal{D}$. This means that we represent each irreducible component of $\mathcal{D}$ by a vertex and we put an edge between two vertices when the corresponding components intersect. We represent the branches of the curve by arrows.



E_2
E_1
E_3
E_4



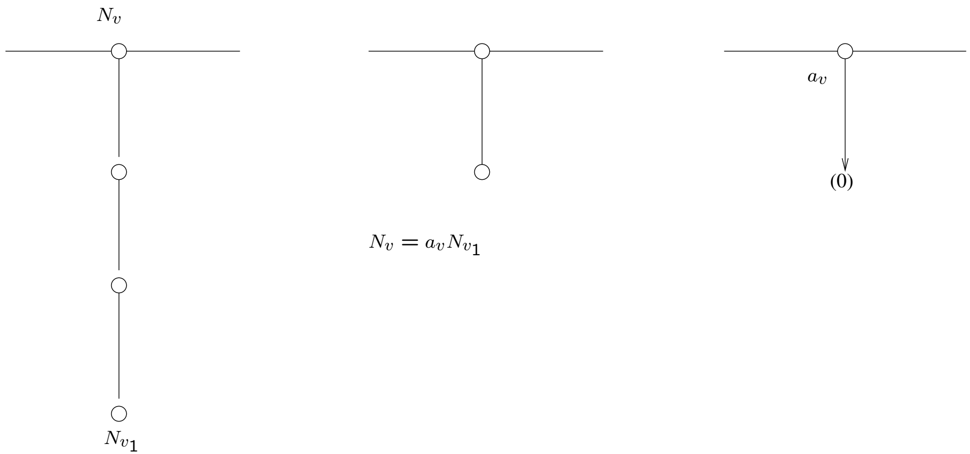
E_1
E_2
E_4
E_3

We call valency of a vertex, the number of edges incident to this vertex.

We keep information of the multiplicity of the vertices.

To simplify the tree, we delete vertices of valency 2, except the one representing $\mathbb{L}_\infty$, (that we call the root, and denote by v_0), and the dicriticals.

We shall assume that the valency of the root is at least 2.



If we have a vertex of valency 1 we replace it by an arrow decorated by 0 so that all vertices of valency 1 are arrows. We call that a dead end.

If v is not a dicritical vertex, we have that

$$N_{v_1} | N_v$$

and we decorate the edge between v and v_1 near v by $a_v = N_v / N_{v_1}$.

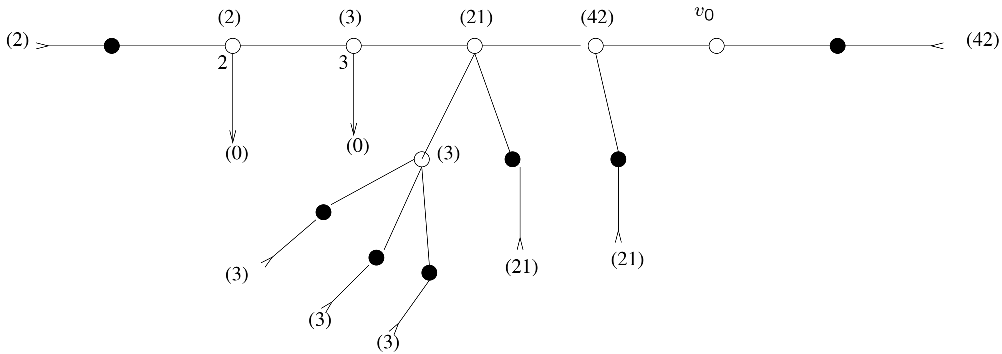
There is at most one dead end attached to a vertex.

Then we end up with a rooted tree.

• Each vertex of valency 1 is an arrow.

• The vertex near an arrow which is not decorated with (0), is a dicritical vertex, whose degree is the number of arrows, not decorated by (0), attached to it.

• Each vertex is decorated by its multiplicity and each dead end is decorated near the vertex v by a positive integer a such that $a|N_v$.

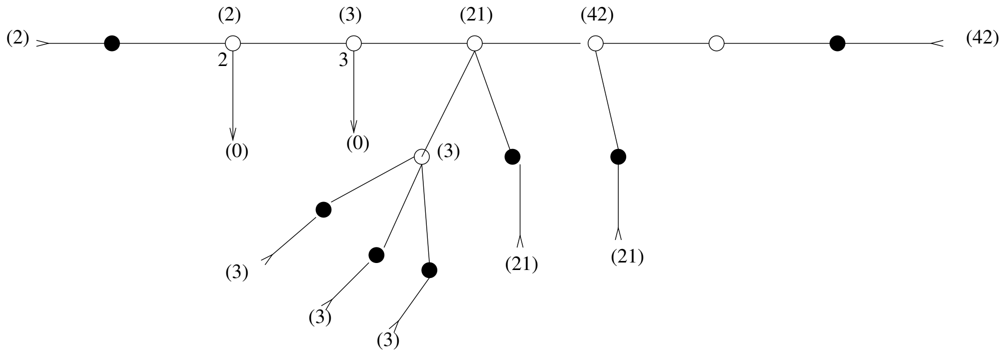


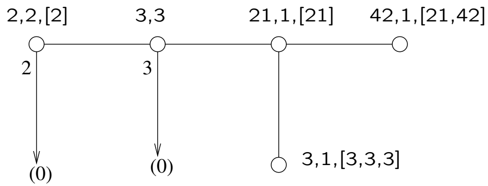
We introduce a partial order on the tree, by saying that $v < v'$ if and only if v is on the path between v_0 and v'.

3: More properties and simplifications of the tree

For each dicritical vertex u there is a unique vertex $v \neq u, v \neq v_0$ such that the path between u and v is linear. We call v the companion of u. A node is a vertex which is not a dicritical and which is the companion of at least one dicritical vertex. The set of nodes is denoted by Nd.

If $v \in Nd$ we denote by $\mathcal{D}_v$ the set of dicriticals attached to v. The type of a node v is the unordered sequence of the degrees of the dicriticals in $\mathcal{D}_v$. From now on we shall not represent the dicriticals, only the nodes, their multiplicities,a_v and the type. (The data $N_v, a_v,$type is called the signature of v).We denote by $\mathcal{N}^*$ the set of vertices that remain when we have deleted the dicritical vertices and eventually the root. $\mathcal{N}^*$ is a tree.


(2)
(2)
(3)
(21)
(42)
(42)
2
3
(0)
(0)
(3)
(3)
(3)
(3)
(3)
(21)
(21)


2,2,[2]
3,3
21,1,[21]
42,1,[21,42]
2
3
(0)
(0)
3,1,[3,3,3]

Proposition 1

$$\forall v \in Nd, \forall u \in \mathcal{D}_v, N_v = k_u d_u$$

We introduce some notation:

1. If $v \in Nd, \sigma(v) = \sum_{u \in \mathcal{D}_v} (k_u - 1) d_u$,

$$d_v = \operatorname*{gcd}_{u \in \mathcal{D}_v} d_u.$$

   If $v \in \mathcal{N}^* \setminus Nd, \sigma(v) = 0$

2. For $v \in \mathcal{N}^*, \epsilon(v) = |\{x \in \mathcal{N}^* | x \text{ is adjacent to } v\}|$.
   It is the valency in $\mathcal{N}^*$.

3. For $v \in \mathcal{N}^*$

$$\tilde{\Delta}(v) = \sigma(v) + (\epsilon(v) - 2)(N_v - 1) + N_v(1 - 1/a_v)$$

Remarks:

1. If $N_v = 1$, then $a_v = 1, k_u = 1 \forall u \in \mathcal{D}_v$, and $\tilde{\Delta}(v) = 0$. Moreover $\epsilon(v) = 1$.

2. If $N_v \neq 1$

   (a) $\epsilon(v) > 2$ implies $\tilde{\Delta}(v) > 0$

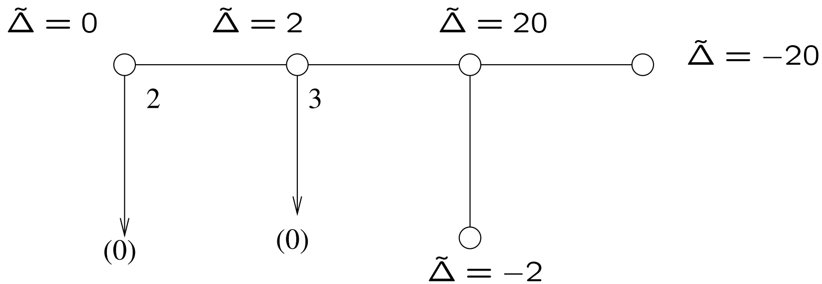
   (b) $\epsilon(v) = 2$ and $\tilde{\Delta}(v) = 0$ implies $N_v, 1, [N_v, \cdots, N_v]$

   (c) $\epsilon(v) = 1$ and $\tilde{\Delta}(v) \leq 0$ implies $N_v, a_v, [d_v, N_v, \cdots,$ $d_v | N_v$ and if $d_v \neq N_v$, then $a_v = 1$.

Let

$$\tilde{\Delta}(\mathcal{N}^*) = \sum_{v \in \mathcal{N}^*} \tilde{\Delta}(v)$$

In the following picture the tree is decorated with $\tilde{\Delta}(v)$



We have

Theorem 2

$$\tilde{\Delta}(\mathcal{N}^*) = 2g$$

where g *is the genus of the generic fiber of* f*.*

We want to study the complexity of the tree in terms of the genus of the generic fiber.

We assume that $\mathcal{N}^*$ has at least 2 elements.

$\mathcal{N}^*$ is the set of vertices that remain when we have deleted the dicritical vertices and eventually the root.

$$\forall v \in Nd, \forall u \in \mathcal{D}_v, N_v = k_u d_u$$

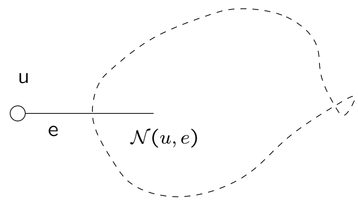
For $v \in \mathcal{N}^*$

$$\tilde{\Delta}(v) = \sum_{u \in \mathcal{D}_v} (k_u - 1) d_u + (\epsilon(v) - 2)(N_v - 1) + N_v(1 - 1/a_v)$$

$$\tilde{\Delta}(\mathcal{N}^*) = \sum_{v \in \mathcal{N}^*} \tilde{\Delta}(v)$$

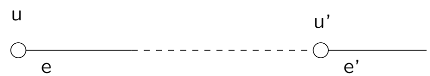
$$\tilde{\Delta}(\mathcal{N}^*) = 2g$$

We introduce more notation:



and $\tilde{\Delta}(u,e) = \sum_{v \in \mathcal{N}(u,e)} \tilde{\Delta}(v)$

We say that $(u', e') < (u, e)$ if the path $\gamma_{u,u'}$ contains e but not e'

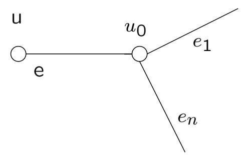


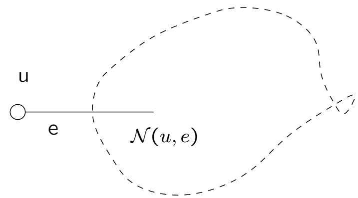
Finally, we introduce the characteristic sequence:

If (u,e) is minimal, i.e. $e = (u, u_0)$, and $\epsilon(u_0) = 1$, we define $c(u,e) = \frac{d(u_0)}{a_{u_0}}$.

If (u,e) is not minimal

$$c(u,e) = \frac{\gcd(d(u_0), c(u_0,e_1), \cdots, c(u_0,e_n))}{a_{u_0}}$$





and $\tilde{\Delta}(u,e) = \sum_{v \in \mathcal{N}(u,e)} \tilde{\Delta}(v)$

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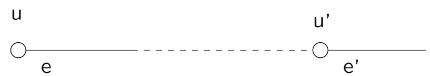


The characteristic sequence $c(u,e)$ is a decreasing sequence defined by induction

4: Last stage of simplification: The skeleton

We say that a path $(z_1, \cdots, z_n)$ is $\tilde{\Delta}$-trivial if for $i = 2, \cdots, n-1$, $\epsilon(z_i) = 2$ and $\tilde{\Delta}(z_i) = 0$. We consider

$$\Gamma = \{\gamma = (z_1, \cdots, z_n), \epsilon(z_1) = 1, \tilde{\Delta}(z_1) \leq 0,$$

$$\tilde{\Delta}(z_n) > 0, z_{n-1} > z_n, \tilde{\Delta} - \text{trivial}\}$$

If $(z_1, \cdots, z_n) \in \Gamma$, we say that $(z_n, (z_n, z_{n-1}))$ is a tooth. We can prove that $c(z_n, (z_n, z_{n-1})) = \frac{d(z_1)}{a_{z_1}}$.

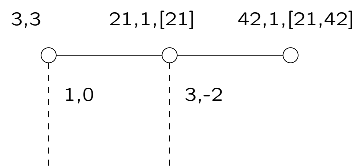
Remarks:

1. If $N_v \neq 1$

   (a) $\epsilon(v) = 2$ and $\tilde{\Delta}(v) = 0$ implies $N_v, 1, [N_v, \cdots, N_v]$

   (b) $\epsilon(v) = 1$ and $\tilde{\Delta}(v) \leq 0$ implies $N_v, a_v, [d_v, \cdots, N_v]$
   $d_v | N_v$ and if $d_v \neq N_v$, then $a_v = 1$.

From $\mathcal{N}^*$ we define the skeleton $\mathcal{S}$: If $(z_1, \cdots, z_n) \in \Gamma$, we delete $z_1, \cdots, z_{n-1}$ and the edges between them and between z_n and z_{n-1}. We keep z_n and the information d_{z_1}/a_{z_1} and $\tilde{\Delta}(z_1)$. $\mathcal{S}$ is a tree.



Description of the Skeleton

Let $\Omega = \{z \in \mathcal{S} | \epsilon(z) = 1, \tilde{\Delta}(z) \leq 0\}$

Proposition 3

$$|\Omega| \leq 2$$

If $|\Omega| = 2$, $\mathcal{N}^* = \{z_1, \cdots, z_n\}$, with
$\epsilon(z_j) = 1, \tilde{\Delta}(z_j) \leq 0, j = 1, n,$
$\epsilon(z_i) = 2, \tilde{\Delta}(z_i) = 0, i = 2, \cdots, n-1$

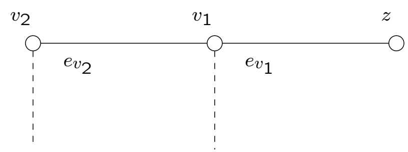
We assume $|\Omega| \leq 1$.

We say that the tree is a brush if $|\mathcal{S}| = 1$.
We assume that the tree is not a brush.

We define $In(\mathcal{S})$ by $In(\mathcal{S}) = \Omega$ when $\Omega \neq \emptyset$, and $In(\mathcal{S}) = \{z \in \mathcal{S} | \delta^*(z) = 1\}$ where δ^* is the valency in $\mathcal{S}$.

We shall describe the skeleton from a $z \in In(\mathcal{S})$.

For $v \in \mathcal{N}^*$, we define e_v as the edge incident to v on the path $\gamma_{v,z}$. We can consider the sequences $c(v, e_v)$ and $\tilde{\Delta}(v, e_v)$. We shall use also $\eta(v, e_v) = \tilde{\Delta}(v, e_v) - (1 - c(v, e_v))$. One can prove that $\eta(v, e_v)$ is an increasing sequence.



$$c(v_1, e_{v_1}) = 21; \tilde{\Delta}(v_1, e_{v_1}) = -20; \eta(v_1, e_{v_1}) = 0$$

$$c(v_2, e_{v_2}) = 3; \tilde{\Delta}(v_2, e_{v_2}) = -2; \eta(v_2, e_{v_2}) = 0$$

We say that (u, e_u) is a comb over $(u', e_{u'})$ if $(u', e_{u'}) <= (u, e_u)$, and $\eta(u, e_u) = \eta(u', e_{u'})$. Remark that (u, e_u) is a comb over itself. If (u, e_u) is a comb over $(u', e_{u'})$, we have for all (v, e_v) such that $(u', e_{u'}) <= (v, e_v) < (u, e_u)$

1. $\epsilon(v) = 2$ and $N_v, a_v, [d_v, N_v, \cdots, N_v], d_v | N_v$ and $d_v \neq N_v$ implies $a_v = 1$

2. $\epsilon(v) = 3$ and $N_v, 1, [N_v, N_v, \cdots, N_v]$ and (v, g) is a tooth.

Let $\mathcal{O} = \{(u, e_u), u \in \mathcal{S}\}$. We define an equivalence relation on $\mathcal{O}$: $(u, e_u) \sim (u', e_{u'})$ if and only if one of (u, e_u) and $(u', e_{u'})$ is a comb over the other. This give us a partition of $\mathcal{O}$ and we denote by $\overline{\mathcal{O}}$ the set of equivalence classes.

Theorem 4 *If the tree is not a brush and* $|\Omega| \leq 1$, *then*

1. $|\overline{\mathcal{O}}| \leq 1 + 2g$

2. *If* $|\overline{\mathcal{O}}| \geq 2g$ *then* $\Omega \neq \emptyset$.

Rational polynomials

A rational polynomial is a polynomial whose generic fiber is a rational curve. They are also called field generators because $f \in k[x, y]$ is a rational polynomial if and only if there exists $g \in k(x, y)$ such that $k(f, g) = k(x, y)$.

Rational polynomials such that all dicriticals have degree 1 (also called simple rational polynomials) are classified:

(Miyanishi and Sugie, Neumann and Norbury, C.N and Daigle).

The case where all dicriticals have degree one except one dicritical with degree 2 has been done by Sasao.

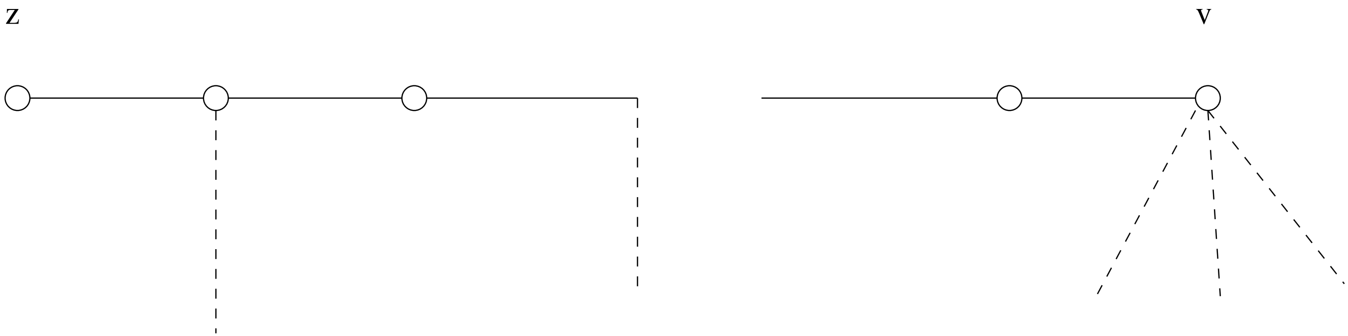
Let f be a rational polynomial. We assume that f is not a variable. We know (Russell) that we can consider trees with valency of the root equal to 2.

<u>Rational trees with $|\mathcal{N}^*| \geq 2$</u>

1. We don't know if there exist rational brushes. What we know is that if they do exist they have at most 3 teeth.

2. $\mathcal{N}^* = \{z_1, \cdots, z_n\}$, with $\epsilon(z_j) = 1, \tilde{\Delta}(z_j) \leq 0, j = 1, n, \epsilon(z_i) = 2, \tilde{\Delta}(z_i) = 0, i = 2, \cdots, n-1$ do exist. Russell's polynomial of degree 21 is an example.

3. **Theorem 5** *Otherwise, we have* $\mathcal{N}^* =$

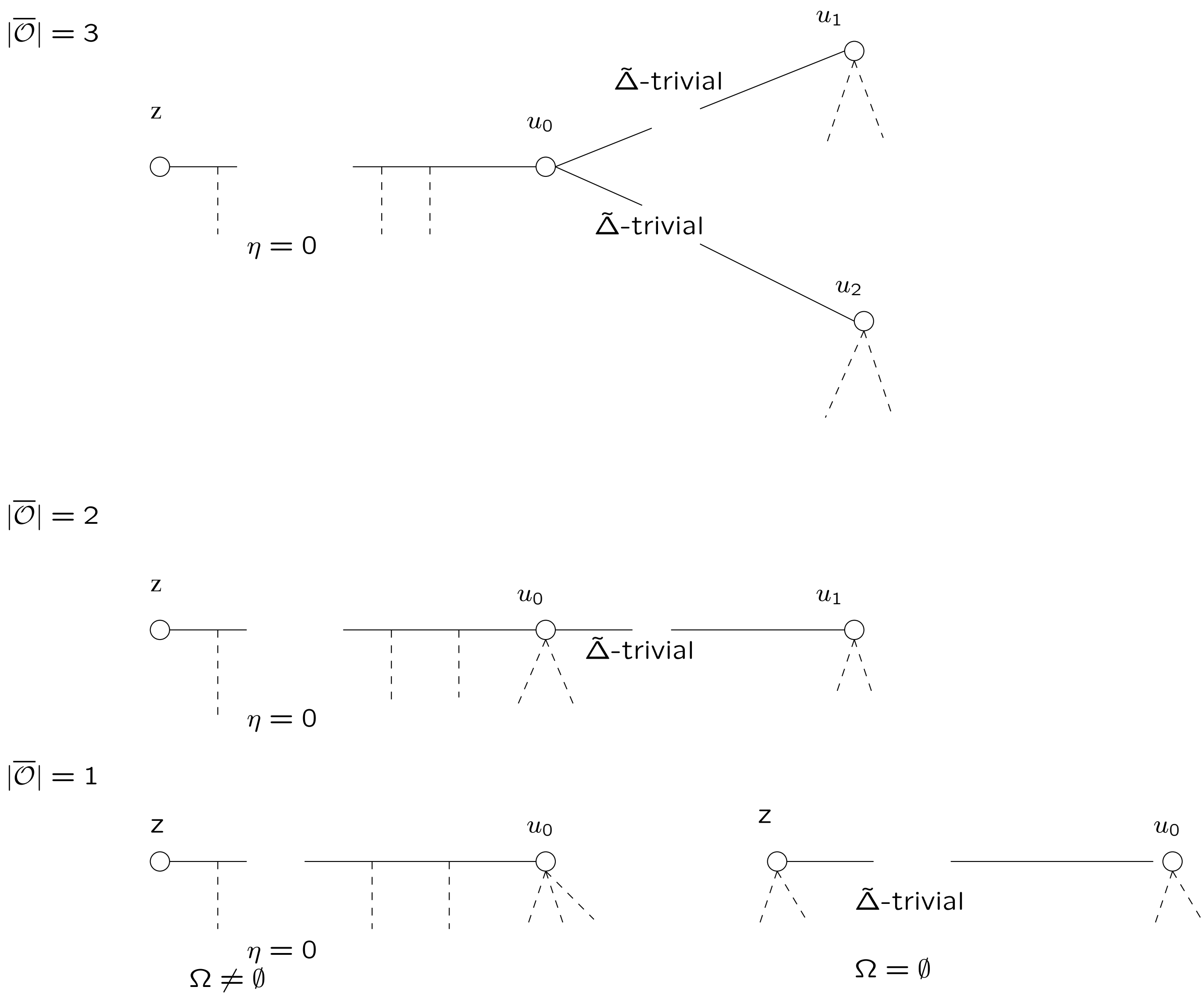


$z \in \Omega$*, one comb with* $\eta = 0$*, and at the end we have at most* 3 *teeth.*

Genus 1 with $|\mathcal{N}^*| \geq 2$

We assume $|\Omega| \leq 1$ and that the tree is not a brush.

We have that $|\overline{\mathcal{O}}| = 1, 2, 3$ and if $|\overline{\mathcal{O}}| \neq 1$, $\Omega \neq \emptyset$.



$|\overline{\mathcal{O}}| = 3$
z
u_0
u_1
u_2
$\tilde{\Delta}$-trivial
$\tilde{\Delta}$-trivial
$\eta = 0$
$|\overline{\mathcal{O}}| = 2$
z
u_0
u_1
$\tilde{\Delta}$-trivial
$\eta = 0$
$|\overline{\mathcal{O}}| = 1$
z
u_0
$\eta = 0$
$\Omega \neq \emptyset$
z
u_0
$\tilde{\Delta}$-trivial
$\Omega = \emptyset$

The signatures of u_0, u_1 and u_2 are not completely determined.