Simple examples of affine manifolds with infinitely many exotic models, Warsaw 2018

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Given any smooth complex affine variety X, one can ask if there exist smooth affine varieties Y non-isomorphic to X but which are biholomorphic to X when equipped with their underlying structures of complex analytic manifold. When such exist, these varieties Y could be called exotic models of X.

Examples of affine varieties with exotic models have been found in dimension two and three by Freudenburg, Moser-Jauslin, Poloni, Dubouloz. Moreover, in our paper [PAMS, 2015] we showed that for every $n \geq 7$ there are n-dimensional rational affine manifolds with exotic models. The aim of this note is to give a simple general method of constructing such examples. In particular we show that examples of affine varieties with uncountably many exotic structures exist in any dimension $d > 1$ (for $d = 1$ it is easy to see that exotic structures do not exist).

Here we prove:

Theorem

Let V be a non-rational smooth affine curve. Then:

(i) The affine surface $Y := V \times \mathbb{C}$ has uncountably many different exotic models.

(ii) For every non $\mathbb{C}$-uniruled smooth affine variety Z the variety $Y \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ of regular automorphisms of $V \times Z$ is at most countable, then the Stein manifold $Y \times Z$ has uncountably many different structures of affine variety.

Remark. If a variety Z is not uniruled, then the group $Aut(V \times Z)$ is automatically finite.

As a consequence we have:

Corollary

Let $\Gamma_1, \ldots, \Gamma_r$ be a finite collection of smooth affine non-rational curves ($r \geq 1$) and let $X = (\prod_{i=1}^{r} \Gamma_i) \times \mathbb{C}$. Then the Stein manifold X has uncountably many different structures of affine variety. In particular for every $d > 1$ there exists a Stein manifold of dimension d which has uncountably many different structures of affine variety.

Moreover we obtain:

Theorem

Let V be a smooth affine surface which has a smooth completion $\overline{V}$ with an effective canonical class (i.e., $H^0(\overline{V}, K_{\overline{V}}) \neq 0$). Then:

(i) The affine fourfold $X := V \times \mathbb{C}^2$ has infinitely many different exotic models.

(ii) For every non $\mathbb{C}$-uniruled smooth affine variety Z the variety $X \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ of regular automorphisms of $V \times Z$ is finite, then the Stein manifold $X \times Z$ has infinitely many different structures of affine variety.

Remark

In particular we can take as V (above) any generic surface $V \subset \mathbb{C}^3$ of degree $d \geq 4$. Moreover, for such V, if a variety Z is not uniruled, then the group $Aut(V \times Z)$ is automatically finite.

Finally, we give the following geometric counterpart of previous Theorem:

Theorem

Let $\overline{V}$ be a smooth projective curve of genus $g > 1$ and let $O \in \overline{V}$ be a sufficiently general point such that $Aut(\overline{V} \setminus \{O\})$ is a trivial group. Take $V = \overline{V} \setminus \{O\}$. Then there exist an algebraic variety $\mathcal{X}$, a g-dimensional torus $\mathbf{T^g}$ and a regular morphism $f : \mathcal{X} \to \mathbf{T^g}$ such that:

1) $f : \mathcal{X} \to \mathbf{T^g}$ is a surjective holomorphic submersion.

2) $f^{-1}(t) \cong V \times \mathbb{C}$ as Stein manifolds for every $t \in \mathbf{T}^g$.

3) $f^{-1}(t) \cong f^{-1}(t')$ as algebraic varieties only if $t = t'$.

Corollary

Let $V = V_1$ *be as above. Let* $V_2, ..., V_r$ *be a finite collection of smooth affine curves such that* $g = g(V) < g(V_2) < ... < g(V_r)$ *and* $Aut(V_i) = \{identity\}$ *for every* $i = 2, ..., r$. *Then there exist an algebraic variety* $\mathcal{X}$, *a* g*-dimensional torus* $\mathbf{T^g}$ *and a regular morphism* $f : \mathcal{X} \to \mathbf{T^g}$ *such that:*

1) $f : \mathcal{X} \to \mathbf{T^g}$ *is a surjective holomorphic submersion.*

2) $f^{-1}(t) \cong (\prod_{i=1}^{r} V_i) \times \mathbb{C}$ *as Stein manifolds for every* $t \in \mathbf{T}^g$.

3) $f^{-1}(t) \cong f^{-1}(t')$ *as algebraic varieties only if* $t = t'$.

Let us recall the definition of a $\mathbb{C}$-uniruled variety. First recall that *a polynomial curve* in an affine variety X is the image of the affine line $\mathbb{C}$ under a non-constant morphism $\phi : \mathbb{C} \to X$. Now we have:

Definition

An affine algebraic variety X is said to be $\mathbb{C}$-*uniruled* if it is of dimension ≥ 1 and there exists a Zariski open, non-empty subset U of X such that for every point $x \in U$ there is a polynomial curve in X passing through x.

Remark

Let us recall that a variety X is uniruled, if it is covered by rational curves. Hence a $\mathbb{C}$-uniruled variety is uniruled (but not conversely).

Let us recall the following:

Lemma

Let X be a smooth affine variety. If $\overline{X}$ is a smooth completion of X and the latter is $\mathbb{C}$-uniruled, then $H^0(\overline{X}, K_{\overline{X}}) = 0$, where $K_{\overline{X}}$ denotes the canonical divisor of $\overline{X}$.

To prove next result we also need the following:

Lemma

Let $f : U \times \mathbb{C}^r \to X$ be a dominant morphism of affine varieties. If there is $u \in U$ such that $\dim f(\{u\} \times \mathbb{C}^r) > 0$ then X is $\mathbb{C}$-uniruled.

Now we have the following generalization of Theorem 3.6 from [J, Math. Ann, 2010]:

Theorem

Let X be a non-$\mathbb{C}$-uniruled smooth affine variety. Let $\mathbf{F}, \mathbf{G}$ be algebraic vector bundles on X of $\mathrm{rank}\ r$. If the total spaces of $\mathbf{F}$ and $\mathbf{G}$ are isomorphic, then $\mathbf{F}$ is isomorphic to $\sigma^*\mathbf{G}$ for some automorphism $\sigma \in Aut(X)$.

Proof. Let F denote the total space of $\mathbf{F}$ and G the total space of $\mathbf{G}$. In what follows, we will identify X with the zero sections of $\mathbf{F}$ and $\mathbf{G}$. Note that

$$\mathbf{F} \cong TF|_X/TX, \ \mathbf{G} \cong TG|_X/TX.$$

Assume that there exists an isomorphism $\Phi : F \to G$. Let $\pi : \mathbf{G} \to X$ be the projection and take $f = \pi \circ \Phi$. Since the variety X is not $\mathbb{C}$-uniruled and the vector bundle $\mathbf{F}$ is locally trivial in the Zariski topology, Lemma above shows that $\Phi(\mathbf{F}_x) = \mathbf{G}_{f(x)}$ for every $x \in X$.

Consequently, $\sigma := f|_X : X \to X$ is a bijection. Moreover, it is easy to check that for every $x \in X$ the derivative $d_x\sigma$ is an isomorphism. Consequently, σ is an automorphism. Take $\mathbf{G}' = \sigma^*\mathbf{G}$. Let $\Sigma : G \to G'$ be the isomorphism of total spaces induced by σ (locally given as $U \times \mathbb{C}^n \ni (x, v) \mapsto (\sigma^{-1}(x), v) \in \sigma^{-1}(U) \times \mathbb{C}^n$). Replace Φ by $\Sigma \circ \Phi$ and $\mathbf{G}$ by $\mathbf{G}'$.

Now the mapping $\Phi|_X : X \ni x \mapsto (x, t(x)) \in G$ is a section. Consider the isomorphism $\Psi : G \ni (x, v) \mapsto (x, v - t(x)) \in G$. Again we can replace Φ by $\Psi \circ \Phi$ to obtain $\Phi|_X : X \times \{0\} \ni (x, 0) \mapsto (x, 0) \in G$. Hence we can assume that Φ transforms the zero section into the zero section, and moreover it induces the identity on the zero section. Hence $d\Phi(TX) = TX$ and the mapping

$$d\Phi : TF|_X/TX \cong \mathbf{F} \to TG|_X/TX \cong \mathbf{G}$$

is an isomorphism. Consequently, the bundle $\mathbf{F}$ is isomorphic to $\mathbf{G}$.$\square$

Now we review some results about complex vector bundles. Topological complex vector bundles on (real) CW complexes have the following nice property:

Theorem

Let Y be an $r-$dimensional CW complex and let $\mathbf{F}$ be a topological complex vector bundle on Y of rank k. If $r \leq 2k-1$, then $\mathbf{F}$ has a one dimensional trivial summand.

Now we make use of Grauert's theorem on the Oka principle for vector bundles which says that on Stein spaces the holomorphic and topological classifications coincide. Therefore we can use the topological theory of complex vector bundles. Moreover, since every n-dimensional Stein manifold has a homotopy type of a (real) n-dimensional CW complex, if we study vector bundles on X, we can assume that X itself is an n-dimensional CW complex. In particular we have:

Corollary

Let Y be an r-dimensional Stein manifold and let $\mathbf{F}$ be a holomorphic vector bundle on Y of rank k. If $r \leq 2k - 1$, then $\mathbf{F}$ has a one-dimensional trivial summand.

Now we are ready to prove our first result:

Theorem

Let V be a non-rational smooth affine curve. Then:

(i) The affine surface $Y := V \times \mathbb{C}$ has uncountably many different exotic models.

(ii) For every non $\mathbb{C}$-uniruled smooth affine variety Z the variety $Y \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ of regular automorphisms of $V \times Z$ is at most countable, then the Stein manifold $Y \times Z$ has uncountably many different structures of affine variety.

(i) Let $\overline{V}$ be a smooth compactification of V and $\{x_1, ..., x_r\} = \overline{V} \setminus V$. Then $Pic(V) = Pic(\overline{V})/\langle x_1, ..., x_r\rangle$. Since the subgroup $\langle x_1, ..., x_r\rangle$ is countable and $Pic(\overline{V})$ is not, we have $Pic(V) \neq 0$. In fact this group is uncountable. Let $\mathbf{L} \in Pic(V)$ be a non-zero line bundle. Hence it is algebraically non-trivial. However, by previuos Corollary, $\mathbf{L}$ is holomorphically trivial. Consequently, the total space of any line bundle $\mathbf{L} \in Pic(X)$ is biholomorphic to Y.

Note that the total space of every line bundle $\mathbf{L} \in Pic(V)$ determines a structure of affine algebraic variety $Y_{\mathbf{L}}$ on Y. Let ρ be the equivalence relation on $Pic(V)$ determined by $\mathbf{L} \sim_{\rho} \mathbf{L}'$ if and only if there exists an automorphism $\sigma \in Aut(V)$ such that $\mathbf{L}' = \sigma^* \mathbf{L}$. Since the group $Aut(V)$ is finite, we see that the set $S := Pic(V)/\rho$ is uncountable. Denote the equivalence class of $\mathbf{L}$ with respect to ρ by $[\mathbf{L}]$.

The structures $Y_{\mathbf{L}}$ and $Y_{\mathbf{L}'}$ are not isomorphic for $[\mathbf{L}] \neq [\mathbf{L}']$ by Theorem on vector bundles. This means that there are at least $\#S$ different affine structures on Y.

(ii) Let $\pi : V \times Z \to V$ be the natural projection. Take $\mathbf{L}' = \pi^*(\mathbf{L})$. Then $\mathbf{L}$ is holomorphically trivial. However, it is algebraically non-trivial. Indeed, take a point $z \in Z$. If we identify V with $V \times \{z\} \subset V \times Z$, then $\mathbf{L}'|_V = \mathbf{L}$.
Note that the mapping $\pi^* : Pic(V) \ni \mathbf{L} \to \pi^*\mathbf{L} \in Pic(V \times Z)$ is injective, hence the group $Pic(V \times Z)$ is uncountable. If the group $Aut(V \times Z)$ is at most countable, then the set $S' = Pic(V \times Z)/\rho$ (where ρ is the equivalence relation as above) is uncountable. Now we can finish as in the point (i).

Corollary

Let $\Gamma_1, ..., \Gamma_r$ *be a finite collection of smooth affine non-rational curves (*$r \geq 1$*) and let* $X = (\prod_{i=1}^{r} \Gamma_i) \times \mathbb{C}$*. Then the Stein manifold* X *has uncountably many different structures of affine variety. In particular for every* $d > 1$ *there exists a Stein manifold of dimension* d *which has uncountably many different structures of affine variety.*

Proof.

Let us note that $\overline{\kappa}(\Gamma_i) = 1$ (where $\overline{\kappa}$ denotes the logarithmic Kodaira dimension). We have $\overline{\kappa}(\prod_{i=1}^r \Gamma_i) = \sum_{i=1}^r \overline{\kappa}(\Gamma_i) = r$. Hence the variety $\prod_{i=1}^r \Gamma_i$ is of general type and consequently it has a finite automorphism group. □

We show that our method can be applied also to affine surfaces. The following lemma is well known:

Lemma

Let X *be a smooth affine surface. Let* $A^p(X)$ *denote the group of codimension* p *cycles modulo rational equivalence. Let* $c_1 \in A^1(X),\ c_2 \in A^2(X)$. *Then there exists an algebraic vector bundle* $\mathbf{F}$ *of rank* 2 *such that* $c_i(\mathbf{F}) = c_i$ *for* $i = 1, 2,$ *where* $c_i(\mathbf{F})$ *is the* i^{th} *Chern class of* $\mathbf{F}$*.*

Moreover, we need the following result of Mumford and Roitman:

Theorem

Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$ with an effective canonical class (i.e. $H^0(X, K_X) \neq 0$, where K_X is the canonical divisor of X). Then for any affine open subset $V \subset X$, we have $A^d(V) \neq 0$.

We also have:

Lemma

Let V be a smooth affine surface which has a smooth completion $\overline{V}$ such that $H^0(\overline{V}, K_{\overline{V}}) \neq 0$. For a given non-zero class $c_2 \in A^2(V)$, there exists an algebraic vector bundle $\mathbf{F}$ of rank two on V which is algebraically non-trivial, but holomorphically trivial and $c_2(\mathbf{F}) = c_2$.

Proof.

By Theorem above we have $A^2(V) \neq 0$. Take a nonzero $c_2 \in A^2(V)$. By Lemma above there is an algebraic vector bundle $\mathbf{F}$ of rank 2 such that $c_1(\mathbf{F}) = 0$ and $c_2(\mathbf{F}) = c_2$. In particular $\mathbf{F}$ is algebraically non-trivial and it has trivial determinant.

Now let us consider $\mathbf{F}$ as a holomorphic vector bundle. We have $\mathbf{F} = \mathbf{L} \oplus \mathbf{E}^1$, where $\mathbf{E}^1$ denotes the trivial line bundle. Since $\mathbf{E}^1 = \wedge^2 \mathbf{F} = \mathbf{L} \otimes \mathbf{E}^1 = \mathbf{L}$ it follows that $\mathbf{F} = \mathbf{E}^1 \oplus \mathbf{E}^1$ is holomorphically trivial. □

Finally we have:

Theorem

Let V be a smooth affine surface which has a smooth completion $\overline{V}$ with an effective canonical class (i.e., $H^0(\overline{V}, K_{\overline{V}}) \neq 0$). Then:

(i) The affine fourfold $X := V \times \mathbb{C}^2$ has infinitely many different exotic models.

(ii) For every non $\mathbb{C}$-uniruled smooth affine variety Z the variety $X \times Z$ has an exotic model. Moreover, if the group $Aut(V \times Z)$ of regular automorphisms of $V \times Z$ is finite, then the Stein manifold $X \times Z$ has infinitely many different structures of affine variety.

Proof. (i) Note that the variety V is not $\mathbb{C}$-uniruled. Let $\mathbf{F}$ be a vector bundle as in Lemma above. This vector bundle is algebraically non-trivial but holomorphically trivial.
Note that the total space of every vector bundle $\mathbf{F}$ as above determines a structure of affine algebraic variety $Y_{\mathbf{F}}$ on Y. Let ρ be the equivalence relation on $A^2(V)$ such that $a \sim_\rho b$ if and only if there exists an automorphism $\sigma \in Aut(V)$ such that $a = \sigma^* b$. Note that the group $A^2(V)$ is infinite, because by Roitman result we have $A^2(V) \otimes \mathbb{Q} \neq 0$.

Since the group $Aut(V)$ is finite, we see that the set $S := A^2(V)/\rho$ is infinite. Denote the ρ-equivalence class of $a \in A^2(V)$ by $[a]$. The structures $Y_{\mathbf{F}}$ and $Y_{\mathbf{F}'}$ are not isomorphic if $[c_2(\mathbf{F})] \neq [c_2(\mathbf{F}')]$. This means that there are at least $\#S$ different affine structures on Y.

(ii) Since V is not $\mathbb{C}$-uniruled, neither is $V \times Z$. Let $\pi : V \times Z \to V$ be the projection. Take $\mathbf{G} = \pi^*(\mathbf{F})$. Then $\mathbf{G}$ is holomorphically trivial. However, it is algebraically non-trivial. Indeed, take a point $z \in Z$. If we identify V with $V \times \{z\} \subset V \times Z$, then $\mathbf{G}|_V = \mathbf{F}$. Note that the mapping $\pi^* : A^2(V) \ni a \mapsto \pi^* a \in A^2(V \times Z)$ is injective, hence the group $A^2(V \times Z)$ is infinite. If the group $Aut(V \times Z)$ is finite, then the set $S' = A^2(V \times Z)/\rho$ (where ρ is the equivalence relation as above) is infinite. Now we can finish as in the point (i).$\Box$

Note that in some cases we have in fact constructed a smooth family of exotic structures parameterized by a smooth complex manifold of positive dimension. This means a holomorphic submersion $f : \mathcal{X} \to S$ of complex manifolds (or rather a smooth regular morphism of smooth algebraic varieties), which represents a holomorphically trivial fiber bundle, such that each fiber $f^{-1}(s) := \mathcal{X}_s$ has a structure of affine algebraic variety and $\mathcal{X}_s = \mathcal{X}_{s'}$ as algebraic varieties only if $s = s'$. In this section we describe this construction explicitly and we show that such families can have arbitrarily large dimension.

Let Γ be a smooth projective curve of genus $g > 0$ and let $Pic_0(\Gamma)$ denote the Jacobian variety of Γ (i.e., $Pic_0(\Gamma)$ parameterizes all divisor classes of degree 0 on Γ). It is well known that $Pic_0(\Gamma)$ has the structure of a g-dimensional algebraic torus $\mathbf{T}^g$. Let us recall the notion of a Poincare line bundle.

Definition

Let Γ *be a smooth projective curve of genus* $g > 0$*. Then there exists a line bundle* $\mathcal{P}$ *on* $\Gamma \times Pic_0(\Gamma)$*, called a Poincare line bundle, such that for every* $\alpha \in Pic_0(\Gamma)$ *we have*

$$\mathcal{P}|_{\Gamma \times \{\alpha\}} \cong \alpha.$$

Let Γ be as above and take a point $O \in \Gamma$. We have:

Lemma

Let $\Gamma' := \Gamma \setminus \{O\}$. *Then* $Pic(\Gamma') = Pic_0(\Gamma)$.

Proof.
Let

$$\Phi : Pic_0(\Gamma) \ni \alpha \mapsto \alpha|_{\Gamma'} \in Pic(\Gamma').$$

Every class $[\sum a_i P_i] \in Pic(\Gamma')$ is the image of the class $[\sum a_i P_i - (\sum a_i)O] \in Pic_0(\Gamma)$, hence the mapping Φ is surjective. Now let $\alpha = [\sum a_i P_i + bO]$ and $\Phi(\alpha) = 0$. Then there is a rational function f on Γ' such that $\sum a_i P_i = (f)$. If $\bar{f}$ is an extension of f to the whole of Γ, then $\sum a_i P_i + bO = (\bar{f})$, i.e., $\alpha = 0$. □

Corollary

Let Γ' *be as above. Then there is a line bundle* $\mathcal{R}$ *on* $\Gamma' \times Pic(\Gamma')$ *(we will call it also a Poincare line bundle) such that*

$$\mathcal{R}|_{\Gamma' \times \alpha} \cong \alpha.$$

Moreover, $Pic(\Gamma')$ *has the structure of a* g*-dimensional torus* $\mathbf{T}^g$*.*

Proof.

Let $\mathcal{P}$ be a Poincare line bundle on $\Gamma \times Pic_0(\Gamma) = \Gamma \times Pic(\Gamma')$. It is enough to take $\mathcal{R} = \mathcal{P}|_{\Gamma' \times Pic(\Gamma')}$. □

Now we are in a position to prove:

Theorem

Let $\overline{V}$ be a smooth projective curve of genus $g > 1$ and let $O \in \overline{V}$ be a sufficiently general point, such that $Aut(\overline{V} \setminus \{O\})$ is a trivial group. Take $V = \overline{V} \setminus \{O\}$. Then there exist an algebraic variety $\mathcal{X}$, a g-dimensional torus $\mathbf{T}^g$ and a regular morphism $f : \mathcal{X} \to \mathbf{T}^{\}}$ such that:

1) $f : \mathcal{X} \to \mathbf{T}^{\}}$ is a surjective holomorphic submersion.

2) $f^{-1}(t) \cong V \times \mathbb{C}$ as Stein manifolds for every $t \in \mathbf{T}^g$.

3) $f^{-1}(t) \cong f^{-1}(t')$ as algebraic varieties only if $t = t'$.

Proof.

Let $\mathcal{R}$ be a Poincare line bundle on $V \times \mathbf{T}^g$ and denote by $\mathcal{X}$ its total space. Let $\pi_1 : \mathcal{X} \to V \times \mathbf{T}^g$, $\pi_2 : V \times \mathbf{T}^g \to \mathbf{T}^g$ be the natural projections and take $f = \pi_2 \circ \pi_1$. Then $f : \mathcal{X} \to \mathbf{T}^g$ is a surjective holomorphic submersion and a smooth regular morphism of algebraic varieties. Note that $f^{-1}(t)$ is the total space of the line bundle $\mathcal{R}|_{V \times \{t\}}$. Hence, by our previous result the mapping f satisfies properties 2) and 3). □

Corollary

Let $V = V_1$ be as above. Let $V_2, ..., V_r$ be a finite collection of smooth affine curves such that $g = g(V) < g(V_2) < ... < g(V_r)$ and $Aut(V_i) = \{identity\}$ for every $i = 2, ..., r$. Then there exist an algebraic variety $\mathcal{X}$, a g-dimensional torus $\mathbf{T}^g$ and a regular morphism $f : \mathcal{X} \to \mathbf{T}^{\}}$ such that:

1) $f : \mathcal{X} \to \mathbf{T}^{\}}$ is a surjective holomorphic submersion.

2) $f^{-1}(t) \cong (\prod_{i=1}^{r} V_i) \times \mathbb{C}$ as Stein manifolds for every $t \in \mathbf{T}^g$.

3) $f^{-1}(t) \cong f^{-1}(t')$ as algebraic varieties only if $t = t'$.

THANK YOU FOR ATTENTION!