# Complete algebraic vector fields on affine surfaces

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## Notations.

In this talk $X$ is always a **normal complex affine algebraic surface** ;

$\bar{X}$ is an SNC-completion of $X$ and $D = \bar{X} \setminus X$ is the boundary divisor ;

$\Gamma$ is the weighted dual graph of $D$, i.e.

the vertices of $\Gamma$ = the irreducible components of $D$

the edges of $\Gamma$ = the double points of $D$ ;

the weight of each vertex is the selfintersection of the

corresponding component of $D$ in $\bar{X}$.

## Definition.

(1) a holomorphic field $\nu$ on $X$ is complete if there is a holomorphic map $\Phi : \mathbb{C} \times X \to X,\ (t, x) \to \Phi(t, x)$ such that $\frac{\partial}{\partial t}\Phi(t, x) = \nu(\Phi(t, x))$. Such $\Phi$ is called the flow of $\nu$.

(2) A complete field $\nu$ is locally nilpotent if its flow is a morphism.

In this case the flow is a $G_a$-action.

(3) A complete field $\nu$ is semi-simple if its flow factors through a morphism $\mathbb{C}^* \times X \to X$.

In this case the flow induces a $G_m$-action.

Brunella classified complete algebraic vector fields on $\mathbb{C}^2$ up to polynomial automorphisms

**Example**. The field $\nu = ay\frac{\partial}{\partial y} + A(x^m y^n)[nx\frac{\partial}{\partial x} - my\frac{\partial}{\partial y}]$ where

$A(t)$ is a polynomial is a complete algebraic vector field on $\mathbb{C}^2_{x,y}$.

## Motivation

Ambitious Aim : classify affine surfaces on which

the group generated by flows of complete

holomorphic vector fields acts homogeneously.

**Definition.** Let a group $G$ act on a normal surface $X$.

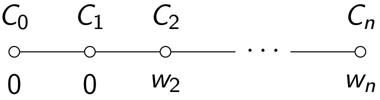
We say that $X$ is $G$-quasi-homogeneous if $G$ has an open orbit whose complement is at most finite.

**Modest Aim**. Classify surfaces quasi-homogenous with respect to reasonable groups

## $\mathrm{Aut}_{\mathrm{alg}}(X)$-Quasi-homogeneous surfaces $X$ (Gizatullin)

Let $X$ be different from $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathbb{C} \times \mathbb{C}^*$

Then $X$ admits an SNC-completion $\bar{X}$ such that the dual graph $\Gamma$ of its boundary $\bar{X} \setminus X$ is a linear rational graph which can be always chosen in the following standard form




where $n \geq 0$ and $w_i \leq -2$ for $i = 2, \ldots, n$.

Gizatullin surfaces are Not necessarily homogeneous (Danilov, Gizatullin and Kovalenko).

**Remark.** $\forall$ Gizatullin surface is $\mathrm{SAut}(X)$-quasi-homogeneous where

$\mathrm{SAut}(X) \subset \mathrm{Aut}_{\mathrm{alg}}(X)$ generated by all elements of $G_a$-actions on $X$

i.e. by the elements of the flows of locally nilpotent vector fields.

**Basic Example**. Let $\delta = x\partial/\partial x + cy\partial/\partial y$ be a vector field on $\mathbb{C}^2$

where $c$ is an irrational number. The the flow of $\delta$ is given by

$$\Phi_t(x, y) = (e^t x, e^{ct} y).$$

Note that $\Phi$ is neither $G_a$-action nor $G_m$-action

and each general integral curve of $\delta$ is everywhere dense in $\mathbb{C}^2$.

## Definition.

An element of a flow of a complete algebraic vector field will be called an algebraically generated holomorphic automorphism.

Let $\mathrm{AAut}_{\mathrm{hol}}(X)$ be the subgroup of $\mathrm{Aut}_{\mathrm{hol}}(X)$ generated by all algebraically generated holomorphic automorphisms.

If $X$ is $\mathrm{AAut}_{\mathrm{hol}}(X)$-quasi-homogeneous we call it generalized Gizatullin surface.

## Main Theorem

$X$ is a generalized Gizatullin surface if and only if for some SNC-completion $\bar{X}$ the divisor $\bar{X} \setminus X$ consists of rational curves, and has a dual graph $\Gamma$ that belongs to one of the following types

(1) a standard zigzag or a linear chain of three 0-vertices (i.e. Gizatullin surfaces and $\mathbb{C} \times \mathbb{C}^*$),

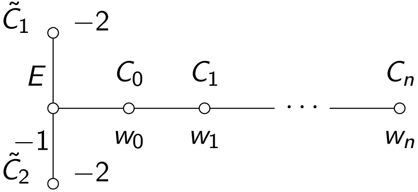
(2) circular graph with the following possibilities for weights

(2a) $((0,0,w_1,\ldots,w_n))$ where $n \geq 0$ and $w_i \leq -2$, and in the case of $n \geq 5$ this cycle is a subgraph of a graph $\tilde{\Gamma}$ contractible to a cycle $((0,0,0,0))$ with all vertices being the proper transforms of the zero vertices in $\Gamma$ and their neighbors

(2b) $((0,0,w))$ with $-1 \leq w \leq 0$ or $((0,0,0,w))$ with $w \leq 0$,

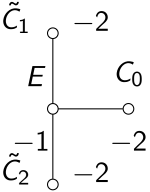
(2c) $((0,0,-1,-1))$;

(3)




where $n \geq 0$, $w_0 \geq 0$ and $w_i \leq -2$ for $i \geq 1$,

(4)



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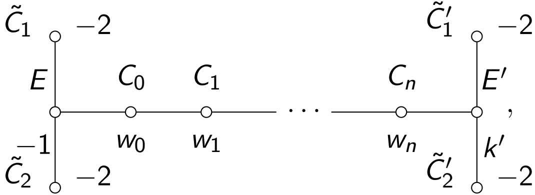


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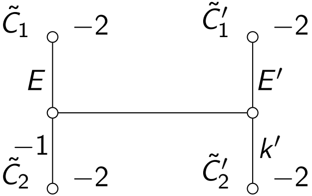
where $n \geq 0, w_0 \geq 0$ and $w_i \leq -2$ for $i \geq 1$;
moreover $k' \leq -1$ if $n = 0$ or $k' \leq -2$ if $n > 0$,

(6)




, for $k' \geq -1$.

**Examples-1** (1) Let $X \subset \mathbb{C}^3$ be a hypersurface given by

$$x + y + xyz = 1.$$

This surface admits an SNC-completion $\bar{X}$ such that the dual graph of $\bar{X} \setminus X$ is a cycle $((0, 0, -1, -1))$

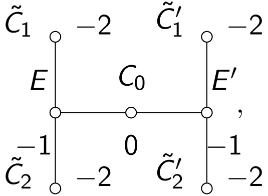
but it has no nontrivial $G_a$ or $G_m$-actions

(2) Let $X \subset \mathbb{C}^3$ be a hypersurface given by

$$xp(x) + yq(y) + xyz = 1$$

where the polynomials $1 - xp(x)$ and $1 - yq(y)$ have simple roots only. It is a generalized Gizatullin surface.

**Example.**

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The Gizatllin surface corresponding to this graph is a twisted $\mathbb{C}^*$-bundle over $\mathbb{C}^*$, i.e. it is a complexification of the Klein bottle.

**Definition.** $\forall$ vector field $\nu$ on $X$ a rational map $f : X \dashrightarrow B$ with $\nu$ tangent to the fibers of $f$ is a rational first integral of $\nu$.

**Theorem B.** Let $X$ admit a nonzero complete algebraic vector field. Then either ($\alpha$) all complete algebraic fields share the same rational first integral, or

($\beta$) $X$ is rational with an open $\mathrm{AAut}_{\mathrm{hol}}(X)$-orbit and $\forall$ complete algebraic vector field $\nu$ on $X$ $\exists$ a regular function $f : X \to B \simeq \mathbb{C}$ with general fibers $\mathbb{C}$ or $\mathbb{C}^*$ and a complete vector field $\mu$ on $B$ for which $f_*(\nu) = \mu$.

**Remark**. For $X = \mathbb{C}^2$ the function $f$ in Theorem B was discovered by Brunella ; $f$ in $(\beta)$ yields a Riccati fibration.

**Example**. Let $\delta = x\partial/\partial x + cy\partial/\partial y$ be a vector field on $\mathbb{C}^2$ where $c$ is irrational. Then $f(x, y) = x$ (resp. $f(x, y) = y$) yields a Riccati fibration.

**Theorem.** (Guillot, Rebelo)

Let $X$ admits a nontrivial complete algebraic vector field $\nu$ and $\bar{\nu}$ be its extension to $\bar{X}$. Then up to a birational transformation of $\bar{X}$ one of the following is true

$(1)$ $\bar{\nu}$ has a rational first integral ;

$(2)$ the field $\bar{\nu}$ is holomorphic ;

$(3)$ $\exists$ a morphism $\bar{f} : \bar{X} \to B$ into a complete rational or elliptic curve $B$ with rational or elliptic general fibers such that $\exists$ a vector field $\mu$ on $B$ for which $\bar{f}_*(\bar{\nu}) = \mu$.

**Step 1**. If (2) holds then either (1) or (3) is true.

**Step 2**. In (3) using Brunella's technique one can exclude Turbulent fibrations $\bar{f}$ (i.e. elliptic fibers of $\bar{f}$) and show that general fibers of $\bar{f}|_X$ is either $\mathbb{C}$ or $\mathbb{C}^*$.

**Step 3**. If $\bar{f}(X) = \mathbb{P}^1$ in (3) then $X$ is a toric surface

. **Step 4**. When $X$ is toric for $\forall$ complete algebraic field $\exists$ $\bar{f}$ as in (3) with $B \simeq \mathbb{C}$.

**Theorem.** Let $X$ be a generalized Gizatullin surface such that for a complete algebraic vector field $\nu$ on $X$ $\exists$ a surjective rational first integral $f : X \dashrightarrow B$ into a complete curve $B$. Then

$(1)$ either $X$ is toric (and, in particular, a Gizatullin surface) or $X$ is isomorphic to the hypersurface $S \subset \mathbb{C}^3_{x,y,z}$ given by

$$y(x^2 + y^2) + z^2 = 0;$$

$(2)$ up to a constant nonzero factor $\nu$ is semi-simple.

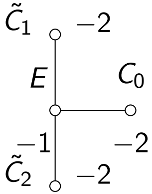
**Remark.** The singularity of $S$ is of type $-D_4$ (Recall that type $-D_{n+1}$

locally isomorphic to the hypersurface $yx^2 + y^n + z^2 = 0$ in $\mathbb{C}^3_{x,y,z}$).

The elliptic $\mathbb{C}^*$-action on $S$ is given by $(x, y, z) \to (\lambda^2 x, \lambda^2 y, \lambda^3 z)$

induced by the field $2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3x\frac{\partial}{\partial z}$ and $\Gamma$ is

(4)



Three $\mathbb{C}^*$-fibrations associated with the three different strings $[[-2,-1,-2]]$ in the graph of $\hat{D}$ are given by the functions

$$y, x+\sqrt{-1}y, \text{ and } x-\sqrt{-1}y.$$

For each of these fibrations $\exists$ a complete vector field tangent to its fibers. Say, for $y$ it is $z\frac{\partial}{\partial x} - xy\frac{\partial}{\partial z}$.

For $n \geq 4$ the hypersurface $\{yx^2 + y^n + z^2 = 0\} \subset \mathbb{C}^3_{x,y,z}$ has an open $\mathrm{AAut}_{\mathrm{hol}}(X)$-orbit but it is not generalized Gizatullin.

**Scheme of the proof in the presence of a Riccati fibration.**

**1**. Let $\bar{X}$ be such that $f$ extends to $\bar{f} : \bar{X} \to \mathbb{P}^1 \implies$

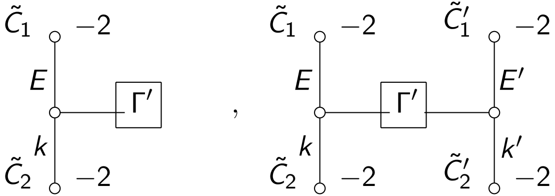
$\exists$ (i) one or (ii) two horizontal components in $D = \bar{X} \setminus X$ and

they are sections, or in (i) it is a double cover of $\mathbb{P}^1$.

**2**. Let $\Gamma$ be pseudo-minimal $\implies$ either the fibers of $\bar{f}$

contained in $D$ are 0-vertices or their graph is $[[-2, -1, -2]]$.

**3**. If $\exists$ another complete vector field then $\exists$ at most two fibers of $\bar{f}$ are contained in $D \implies \Gamma$ with a branch vertex is of the form




**4**. $\implies$ the desired form.

**Concluding Remark.** All Kovalenko's examples of non-homogeneous Gizatullin surfaces $X$ are homogeneous with respect to $\mathrm{AAut}_{\mathrm{hol}}(X)$.