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Normal log canonical del Pezzo surfaces of rank one (j.w.w. Takeshi Takahashi)

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1 Introduction

Minimal model theory for normal surfaces

· smooth case

· with only l.t. singular points

· $(/\mathbb{C})$ normal Moishezon surfaces (Sakai '85)

· with only $\mathbb{Q}$-factorial singular points (Fujino '12, Tanaka '14)

· with only l.c. singular points (Fujino '12, Tanaka '14)

(l.t. $=$ log terminal, l.c. $=$ log canonical)

$/\mathbb{C}$
V: a normal projective surface with only l.c. singular points
$f : V \to W$: a minimal model program for V

f is constructed by contracting curves C with $C^2 < 0$ and $CK_V < 0$ successively.

Theorem 1.1. (Fujino '12, Tanaka '14)

(1) W *is a normal projective surface with only l.c. singular points.*

(2) *One of the followings holds.*

(i) K_W *is nef.* (W *is a minimal model.*)

(ii) *There exists a fibration* $\pi : W \to T$ *onto a smooth projective curve* T *whose general fiber* $\cong \mathbb{P}^1$*.*

(iii) $K_W^2 > 0$*,* $(-K_W)C > 0$ *for any irreducible curve* C *on* W*, and* $\rho(W) = 1$*.*

We call the surface W in Theorem 1.1 (2) (iii) a normal l.c. del Pezzo surface of rank one. If W has only l.t. singular points, then it is called a log del Pezzo surface of rank one.

Remark 1. Sakai's results on ruled fibrations on normal surfaces ('87)
$\Longrightarrow$ Theorem 1.1 (2) (ii)

2 Some results on normal del Pezzo surfaces

X : a normal del Pezzo surface/$\mathbb{C}$

i.e.,

$\cdot$ X: a normal complete algebraic surface

$\cdot$ $K_X^2 > 0$, $(-K_X)C > 0$ for $\forall C$: irreducible curve on X

X is said to be of rank one if $\rho(X) = 1$.

Remark 2. $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$.

Proposition 2.1. (Brenton '77, Chel'tsov '97 etc.)

(1) X *is projective.*
(2) X *is birationally ruled.*
(3) X *is a rational surface.*
$\iff \forall$*singular point on* X *is rational.*
$\iff X$ *is* $\mathbb{Q}$*-factorial.*

Some results

· Normal del Pezzo surfaces with only Gorenstein singular points (Demazure '77, Hidaka–Watanabe '81, etc.)

· Chel'tsov ('97) classified the normal del Pezzo surfaces of rank one with non-rational singular points.

From now on, we consider normal del Pezzo surfaces with only rational singular points.

Some results on log del Pezzo surfaces

· Boundedness of log del Pezzo surfaces (Nikulin '89, '90)

Theorem 2.2. (Gurjar-Zhang '94, '95, Fujiki–Kobayashi–Lu '93) *The fundamental group of the smooth part of a log del Pezzo surface is finite.*

Theorem 2.3. (Keel–McKernan '99) *The smooth part of a log del Pezzo surface is log uniruled.*

· Low indices (classification)
Index two: Alexeev–Nikulin '89, Nakayama '07
Index three: Fujita–Yasutake '17

Theorem 2.4. (Belousov '08) *The number of the singular points on a log del Pezzo surface of rank one* ≤ 4.

· D.-S. Hwang ('14 (?)) announced a classification of the log del Pezzo surfaces of rank one with 4 singular points.

3 Normal log canonical del Pezzo surfaces of rank one

Setting:

X: a normal del Pezzo surface of rank one with only rational l.c. singular points

$\pi : V \to X$: the minimal resolution of X

D: the reduced exceptional div. w.r.t. π

Theorem 3.1. (K.-T. '12) $\# \operatorname{Sing} X \leq 5$.

Theorem 3.2. (K. '13) *Assume* $\# \operatorname{Sing} X = 5$.

(1) *The weighted dual graph of D is given in Fig. 1.*

(2) *There exists a $\mathbb{P}^1$-fibraton $\Phi : V \to \mathbb{P}^1$ in such a way that the configuration of D as well as all singular fibers of Φ can be described in Fig. 2, where a dotted line (resp. a solid line) stands for a (-1)-curve (resp. a component of D).*

Proposition 3.3. (K.-T. in preperation) *X has at most one non l.t. singular point.*

Theorem 3.4. (K.-T. in preperation) *Assume that $\# \operatorname{Sing} X = 4$ and X has a non l.t. singular point. Then there exists a $\mathbb{P}^1$-fibraton $\Phi : V \to \mathbb{P}^1$ such that $FD = 1$ for a fiber F of Φ.*

$\Longrightarrow$ Classification

Minimal compactifications of $\mathbb{C}^2$

X: a minimal compactification of $\mathbb{C}^2$

$\cdot$ X is a normal compact complex surface,

$\cdot$ $\exists\Gamma$: an irreducible closed subvariety on X s.t.
$X \setminus \Gamma$ is biholomorphic to $\mathbb{C}^2$

$\pi : V \to X$: the minimal resolution of X

D: the reduced exceptional divisor on V

C : the proper transform of Γ on V

Theorem 3.5. (K. '00,, K.-T. '09) *Assume that X has only l.c. singular points*

(1) *X is a normal del Pezzo surface of rank one and the compactification (X, Γ) of $\mathbb{C}^2$ is algebraic.*
(2) *The dual graph of D is given in Fig. 3.*

Theorem 3.6. (K.-T. '09, in preperation)
X: a normal del Pezzo surface of rank one with only rational l.c. singular points.
Assume:
· X has a non l.t. singular point or a non-cyclic quotient singular point.
· The singularity type of X is one of the list of Fig. 3.

Then X contains $\mathbb{C}^2$ as a Zariski open subset.

4 On proofs

Setting:

X: a normal del Pezzo surface of rank one with only rational singular points

$\pi : V \to X$: the minimal resolution of X

D: the reduced exceptional div. w.r.t. π

$D^{\#} := \pi^*(K_X) - K_V$

$\mathrm{MV}(X)$: the set of all irreducible curves C' s.t. $C'(-K_X)$ attains the smallest value.

$\mathrm{MV}(V, D) := \{\pi'(C') \mid C' \in \mathrm{MV}(X)\}$

Definition 1. (1) X (or (V,D)) is of the first kind $\iff \exists C \in \mathrm{MV}(V,D)$ s.t. $|C+D+K_V| \neq \emptyset$.
(2) X (or (V,D)) is of the second kind $\iff$ it is not of the first kind, i.e., $\forall C \in \mathrm{MV}(V,D)$, $|C+D+K_V| = \emptyset$.

Remark 3. If $|C+D+K_V| = \emptyset$, then $C+D$ is an SNC-div. and every connected component of $C+D$ is a tree of $\mathbb{P}^1$'s.

Lemmas 4.1∼4.4, 4.6 are obtained by Miyanishi and Zhang when X has only l.t. singular points.

Lemma 4.1. *With the same notations and assumptions as above, we have:*

(1) $-(D^{\#}+K_V)$ *is nef and big* $\mathbb{Q}$*-Cartier divisor.*

(2) F*: an irreducible curve* $-F(D^{\#}+K_V)=0$ *if and only if* F *is a component of* D*.*

(3) *Any* $(-n)$*-curve with* $n \geq 2$ *is a component of* D*.*

Lemma 4.2. *If X is of the first kind, then there exists uniquely a decomposition of D as a sum of effective integral divisors $D = D' + D''$ s.t.*

(i) $CD_i = D''D_i = K_V D_i = 0$ *for any irreducible component D_i of D'.*

(ii) $C + D'' + K_V \sim 0$.

In particular, X has only l.t. singular points.

We assume that X is of the second kind and $\rho(V) \geq 3$.

Lemma 4.3. *Every curve* $C \in \mathrm{MV}(V, D)$ *is a* (-1)*-curve.*

Lemma 4.4. *Let* $C \in \mathrm{MV}(V, D)$ *and let* $D_1, \ldots, D_r$ *be the components of* D *meeting* C*. Then* $\{-D_1^2, \ldots, -D_r^2\}$ *is one of the following:* $\{2_a, n\}$, $\{2_a, 3, 3\}$, $\{2_a, 3, 4\}$, $\{2_a, 3, 5\}$, *where* 2_a *signifies that* 2 *is repeated* a*-times.*

We also use the following:

Lemma 4.5. $C' \in \mathrm{MV}(X)$.
$\Longrightarrow X \setminus C'$ *is a* $\mathbb{Q}$*-homology plane.*

· Palka ('13) classified the $\mathbb{Q}$-homology planes containing at least one non l.t. singular points.

$C \in \mathrm{MV}(V, D)$
$D_1, \ldots, D_r$: the components of D meeting C.

Case (II-1) $r \geq 2$ and $D_1^2 = D_2^2 = -2$. (Type II-1)
Case (II-2) $r = 1$ ($\Longrightarrow D_1^2 = -2$). (Type II-2)
Case (II-3) $r = 3$ and $\{D_1^2, D_2^2, D_3^2\} = \{-2, -3, -3\}$, $\{-2, -3, -4\}$ or $\{-2, -3, -5\}$. (Type II-3)
Case (II-4) $r = 2$ and $D_1^2 \leq -3$. (Type II-4)

We assume further that:

$\cdot$ X has only rational l.c. singular points.

$\cdot$ X has at least one non l.t. singular point.

($\Longrightarrow X$ is of the second kind.)

Case (II-1) (K.-T. '12)

D_1, D_2: two (-2)-curves $\subset \operatorname{Supp} D$ meeting C.

$|C + D + K_V| = \emptyset \Longrightarrow CD_1 = CD_2 = 1$,

$D_1 D_2 = 0$

$\Longrightarrow |D_1 + D_2 + 2C|$ gives rise to a $\mathbb{P}^1$-fibration

$\Longrightarrow$ Classification of the pairs (V, D).

Case (II-2) (K.-T. In preparation)

$D^{(1)}$: the connected component of D meeting C

By case by case study on the shape of the dual graph of $D^{(1)}$ and by using the result of Palka, we can determine the pairs (V, D).

However, we do not have to use this classification in proofs of the results of §3.

Case (II-3) (K.-T. In preparation)
$C \in \mathrm{MV}(V, D)$
D_1, D_2, D_3: the three irreducible components of D meeting C
$G := 2C + D_0 + D_1 + D_2 + K_V$

Lemma 4.6. *Either $G \sim 0$ or there exists a (-1)-curve Γ such that $G \sim \Gamma$ and $\Gamma C = \Gamma D_i = 0$ for $i = 0, 1, 2$.*

By using Lemma 4.6 and the result of Palka, we can determine the pair (V, D).

Case (II-4) ???

We can prove the results of §3 by using some results on $\mathbb{Q}$-homology planes.

On the proof of Theorem 3.4

$D^{(1)}$: the connected component of D that is contracted to a non l.t. singular point
D_0: a branch comp. of $D^{(1)}$.

Lemma 4.7. $(V, D - D_0)$ *is almost minimal and* $\overline{\kappa}(V - (D - D_0)) = -\infty$*.*