Affine space fibrations

M. Miyanishi
jointly with
R.V. Gurjar and K. Masuda

RCMS, Kwansei Gakuin University

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Contents of the talk

$k = \overline{k}$: the ground field of characteristic zero. A dominant morphism $f : X \to Y$ of algebraic varieties is a fibration equivalently if

- the generic fiber X_η is geometrically integral,
- general fibers are integral,
- $k(X) \supset k(Y)$ is a regular extension,
- $k(Y)$ is algebraically closed in k(X).

Let F be an algebraic variety. We say that f is an F-fibration if X_y is isomorphic to F over k for general (closed) points y of Y. The F-fibration is locally trivial if $f^{-1}(U) \cong_U U \times F$ for an open set $U \neq \emptyset$ of Y. f is locally isotrivial if $f^{-1}(U) \times_U U' \cong_{U'} U' \times F$ for an open set U and a finite étale covering $U' \to U$. If f is locally trivial (or locally isotrivial) for an open neighborhood U_y (or a étale finite covering U'_y of U_y) of each closed point y then X is an F-bundle over Y (or an étale F-bundle over Y). Let $X_\eta := X \times_Y \mathrm{Spec}\, k(Y)$ be the generic fiber of f. Then $X_\eta \cong_{k(Y)} F$ (or $X_\eta \cong_K F$ for an algebraic extension $K/k(Y)$) if f is locally trivial (or locally isotrivial). A fundamental question is if a fibration $f : X \to Y$ is locally trivial or locally isotrivial provided $X_y \cong F$ for gneral points $y \in Y$.

The generic isotriviality of an F-fibration for an affine variety F follows from

Gereric Equivalence Theorem of Kraft-Russell

Let k be an algebraically closed field of infinite transcendence degree over the prime field. Let $p : S \to Y$ and $q : T \to Y$ be two affine morphisms where S, T and Y are k-varieties. Assume that for all $y \in Y$ the two (schematic) fibers $S_y := p^{-1}(y)$ and $T_y := q^{-1}(y)$ are isomorphic. Then there is a dominant morphism of finite degree $\varphi : U \to Y$ and an isomorphism $S \times_Y U \cong_U T \times_Y U$.

If $f : X \to Y$ is an F-fibration, the generic fiber X_η is a $k(U')/k(Y)$-form of F. The triviality of the form X_η is the generic local triviality of f. If $F = \mathbb{A}^n$, an F-fibration is an affine space fibration. The local triviality is a Dolgachev-Weisfeiler problem. If $n = 1, 2$ the answer is affirmative.

An F-fibration $f : X \to Y$ has singular fiber which is, by definition, a closed fiber X_y which is not isomorphic to F. There are four possibilities for which the fiber X_y is not isomorphic to F.

(1) The fiber X_y is integral, but not isomorphic to F.

(2) The fiber X_y is not integral. Hence either X_y has two or more irreducible components (reducible fiber), or X_y is irreducible but non-reduced (non-reduced fiber).

(3) Each irreducible component Z_i of X_y has right dimension $\dim F$, but has multiplicity $\mathbf{length}\,\mathcal{O}_{X_y,\xi_i} > 1$ which is a multiple of some integer $d \geq 1$ (multiple fiber), where ξ_i is the generic point of Z_i.

(4) Some irreducible component Z_i has dimension bigger than $\dim F$.

Let $f : X \to C$ be an $\mathbb{A}^1$-fibration on an algebraic surface X and let F be a fiber of f. To study F, the following reduction is possible.

(1) The curve C is smooth with C replaced by the normalization $\widetilde{C}$ and X by $X \times_C \widetilde{C}$.

(2) f is an affine morphism with C replaced by an affine open nbd U of $P := f(F)$ and X by $f^{-1}(U)$.

(3) f is the quotient morphism by a G_a-action on X.

Then our problem is :

Problem 1.1

Is every fiber $F := f^{-1}(P)$ for $P \in C$ a disjoint union of the affine lines?

Our present knowledge is:

Theorem 1.2

(1) F is a disjoint union of the irreducible components, each of which is an affine rational curve with one place at infinity.

(2) If an irreducible component Z_i of F is reduced in F, then $Z_i \cong \mathbb{A}^1$.

(3) If X is normal, F is a disjoint union of the affine lines. Every singular point on X is a cyclic quotient singularity.

Theorem 1.3

Let $f : X \to C$ be a dominant morphism from an affine surface X to an affine curve C. Assume that, for every closed point $P \in C$, the fiber $f^{-1}(P)$ is a disjoint union of the affine lines. Then there exists a nontrivial G_a-action on X such that the morphism f is factored by the quotient morphism $q : X \to X/G_a$ as $f : X \xrightarrow{q} X/G_a \xrightarrow{g} C$, where g is a quasi-finite morphism.

Let $f : X \to C$ be a $\mathbb{P}^1$-fibration from an algebraic surface to an algebraic curve. We may assume that C is normal. The generic local triviality follows from Tsen's Theorem.

Lemma 2.1

Suppose that X is normal. Then we have:

(1) X has only rational singularities, whose resolution graph is a tree of smooth rational curves and is a part of a degenerate fiber of a $\mathbb{P}^1$-fibration on a smooth surface.

(2) Every fiber F of f is a union of smooth rational curves, and its intersection dual graph is a tree in the sense that the dual graph of the inverse image of F in a minimal resolution of singularity of S is a tree.

(3) $H_1(F; \mathbb{Z}) = 0$.

Lemma 2.2

Let $f : X \to C$ be a $\mathbb{P}^1$-fibration over a normal curve C. Let F be a fiber of f. Then we have.

(1) The singular locus of X is contained in the union of finitely many fibers of f.

(2) F is a connected union of rational irreducible components.

(3) $\pi_1(F)$ is a cyclic group. If X is normal, F is simply-connected.

Lemma 2.3

Let $f : X \to C$ be a projective morphism which is a $\mathbb{P}^1$-fibration over a smooth curve C and let F be a fiber of f. Then F is simply connected. In particular, ecery irreducible component is homeomorphic to $\mathbb{P}^1$.

One of the natural looking but hard to prove results is:

Theorem 3.1

Let X be a smooth affine threefold with a G_a-action and let $q : X \to Y$ be the quotient morphism. Assume that Y is smooth. Let $F = q^{-1}(P)$ be a fiber and write $F = \Gamma + \Delta$, where Γ (resp. Δ) is pure 1- (resp. 2-) dimensional. Then we have:

(1) Γ is a disjoint union of $\mathbb{A}^1$s and $\Gamma \cap \Delta = \emptyset$.

(2) Let S be a component of Δ. Let L be a general hyperplane section of Y through the point P and let T be the closure in X of $q^{-1}(L \setminus \{P\})$. Assume that $T \cap S \neq \emptyset$. Then S has an $\mathbb{A}^1$-fibration parallel with f.

Turning to a $\mathbb{P}^1$-fibration $f : X \to Y$ with $\dim X = n$ and $\dim Y = n - 1$, we first note that f is not necessarily generically locally trivial if $n \geq 3$.

Lemma 3.2

Suppose that X and Y are smooth. Then we have:

(1) Let S be the closure of the set of points $Q \in Y$ such that either the $F_Q := X \times_Y \mathrm{Spec}\, k(Q)$ has an irreducible component of dimension > 1 or every irreducible component F_i of F_Q has multiplicity > 1, i.e., $\mathrm{length}\, \mathcal{O}_{F_Q,F_i} > 1$. Then $\mathrm{codim}_Y S > 1$.

(2) Let $n = 3$. Then every fiber F_Q is simply-connected.

(3) Let $n = 3$ and write $F_Q = \Gamma + \Delta$ as in the case of $\mathbb{A}^1$-fibration. Then $H_1(\Delta; \mathbb{Z}) = H_1(\Gamma; \mathbb{Z}) = 0$. Each component of Γ is a rational curve, and each component of Δ is a rational surface or a rationally ruled surface.

If $n = 2$ we have:

Lemma 3.3

Let $f : X \to C$ be a $\mathbb{P}^1$-fibration over a smooth curve C and let F be a singular fiber of f. Then F is simply connected. In particular, ecery irreducible component is homeomorphic to $\mathbb{P}^1$.

In general, we have:

Theorem 3.4

Let $f : X \to Y$ be an equi-dimensional $\mathbb{P}^1$-fibration over a smooth variety Y and let F be a closed fiber of f. Then F is simply-connected.

Equivariant setting of Abhyankar-Sathaye conjecture

Let $A = k[x,y,z] = k^{[3]}$, δ a nonzero lnd, $B = \mathrm{Ker}\,\delta$, $f \in B$ such that $X_0 := \{f = 0\} \subset X := \mathrm{Spec}\,A$ is isomorphic to $\mathbb{A}^2$. Is $X_c = \{f = c\}$ isomorphic to $\mathbb{A}^2$ for every $c \in k$? Namely, is $f : X \to C := \mathrm{Spec}\,k[f] \cong \mathbb{A}^1$ an $\mathbb{A}^2$-bundle ?

Theorem 4.1

Assume the following conditions:

(1) The ideal $(f - c)B$ does not contain the plinth ideal $\mathrm{pl}\,(\delta) = B \cap \delta(A)$ for every $c \in k$.

(2) The affine domain A/fA is normal.

Then f is a coordinate of B, i.e., f is an $\mathbb{A}^2$-bundle.

The condition (1) enables us to use the local slice construction in the fiber X_c of the morphism $f : X \to C := \mathrm{Spec}\,k[f]$.

The morphism f is the composite $f : X \xrightarrow{q} Y := X//G_a \xrightarrow{p} \mathbb{A}^1$, where p is induced by the inclusion $k[f] \hookrightarrow B$. Our proof will consist of showing that:

1. The curve $Y_c = \operatorname{Spec} B/(f-c)B$ in Y is the affine line in the affine plane Y.
2. The restriction of the quotient morphism $q|_{X_c} : X_c \to Y_c$ is an $\mathbb{A}^1$-bundle.

The induced G_a-action on X_c is non-trivial. Hence $q|_{X_c}$ is decomposed as $q|_{X_c} : X_c \xrightarrow{q_c} X_c//G_a \xrightarrow{p_c} Y_c$, where q_c is the quotient morphism.

Lemma 4.2

For every $c \in k$, the element $f - c$ is irreducible in B.

Let $\mathbf{Sing}\,(q)$ be the locus of singular fibers of q. Then $\mathbf{Sing}\,(q)$ is a closed set of Y.

Lemma 4.3

The curve $Y_0 = \mathbf{Spec}\, B/fB$ is isomorphic to $\mathbb{A}^1$, and $Y_0 \cap \mathbf{Sing}\,(q) = \emptyset$.

We show that $p_0 : X_0 /\!/ G_a \to Y_0$ is isomorphic by the conditions (1) and (2). Then q_0 is an $\mathbb{A}^1$-bundle.

Corollary 4.4

(1) For every element $c \in k$, the curve Y_c is isomorphic to $\mathbb{A}^1$.

(2) $\mathbf{Sing}\,(q)$ is either the empty set or a finite disjoint union $\coprod_i Y_{c_i}$ with $c_i \in k$.

(3) For every $c \in k$, $p_c : X_c /\!/ G_a \to Y_c$ is an isomorphism and q_c is an $\mathbb{A}^1$-bundle.

(4) $\mathbf{Sing}\,(q) = \emptyset$. Hence $q : X \to Y$ is an $\mathbb{A}^1$-bundle.

Let $X = \operatorname{Spec} A$ be an affine K-variety of dimension n with $K \supset k$ such that $X \otimes_K \overline{K} \cong_{\overline{K}} \mathbb{A}^n$, where $\overline{K}$ is an algebraic closure of K. Suppose that X has a proper action σ of a commutative unipotent K-group G of dim $n-2$. Let $B = A^G$ and $Y = \operatorname{Spec} B$.

Lemma 5.1

Suppose that $q : X \to Y$ has a cross-section if $n \geq 5$. Then:

(1) $Y \cong_K \mathbb{A}^2$.

(2) The graph morphism $\Psi_{\overline{X}/\overline{Y}} := (\overline{\sigma}, p_2) : G \times \overline{X} \to \overline{X} \times_{\overline{Y}} \overline{X}$ is surjective. Further, the generic fiber $X_\eta := X \times_Y \operatorname{Spec} k(Y) \cong_{k(Y)} \mathbb{A}^{n-2}$.

(3) The $\overline{G}$-action on $\overline{X}$ is fixed-point free, and each fiber of $\overline{q}$ is a $\overline{G}$-orbit, hence isomorphic to $\mathbb{A}^{n-2}$ if considered with reduced structure.

(4) For each closed point P of Y, the fiber $q^{-1}(P)_{\mathrm{red}}$ is isomorphic to $\mathbb{A}^{n-2}$.

Lemma 5.2

$\overline{q}$ has no singular fibers over codimension one points of $\overline{X}$. Hence all singular fibers are either empty fibers or irreducible multiple fibers whose reduced form is isomorphic to $\mathbb{A}^{n-2}$. Let $\overline{S}$ be the set of points $\overline{P}$ of $\overline{Y}$ such that $\overline{q}^{-1}(\overline{P})$ is either the empty set or a singular fiber of $\overline{q}$. Then $\overline{S}$ is a finite set.

Lemma 5.3

The quotient morphism $\overline{q} : \overline{X} \to \overline{Y}$ has no multiple fibers. Hence the set $\overline{S}$ consists of the points of $\overline{Y}$ over which the fiber is the empty set.

Lemma 5.4

Let $\overline{Y}_0 := \overline{Y} \setminus \overline{S}$ and let $\overline{X}_0 := \overline{q}^{-1}(\overline{Y}_0)$. Then $\overline{X}_0 = \overline{X}$ and $\overline{X}_0$ is a G-torsor over $\overline{Y}_0$.

Theorem 5.5

Let $X = \operatorname{Spec} A$ be a K-form of $\mathbb{A}^n$ with a proper G-action, where G is a commutative unipotent group of dimension $n-2$. If $n \geq 5$ we suppose that $q : X \to Y//G$ has a cross-section. Then $X \cong_K \mathbb{A}^n$.

Since $\overline{X}$ is a $\overline{G}$-torsor over $\overline{Y}_0$. We may assume that $\overline{K} = \mathbb{C}$. By a long exact sequence of homotopy groups for a fiber bundle, we have $\pi_i(\overline{Y}_0) \cong \pi_i(\overline{X}) \cong \pi_i(\mathbb{A}^n) = 0$ for every $i > 0$. Since $\pi_1(\overline{Y}_0) = (1)$, we have $H_i(\overline{Y}_0; \mathbb{Z}) \cong H_i(\overline{X}; \mathbb{Z}) = 0$ for every $i > 0$ by Hurwicz's isomorphism theorem. But, if $\overline{S} \neq \emptyset$, then $H_3(\overline{Y}_0; \mathbb{Z}) \cong \mathbb{Z}^{\oplus \#(\overline{S})} \neq 0$. This implies $\overline{S} = \emptyset$. Hence X is a G-torsor over Y. Since G is unipotent, there exists a normal subgroup G_1 such that $G_1 \cong G_a$ and G/G_1 is unipotent. Let $X_1 := X//G_1$. Then X is a G_a-torsor over X_1 and X_1 is a G/G_1-torsor over Y. Since $Y \cong_K \mathbb{A}^2$ and G/G_1 has a central normal series whose subquotients are G_a, $X_1 \cong_K \mathbb{A}^{n-1}$ by induction, and $X \cong_K G_a \times X_1$. Hence $X \cong_K \mathbb{A}^n$.

Let X be a K-form of $\mathbb{A}^n \times \mathbb{A}^1_*$. Let $C = \mathrm{Spec}\, R$ with $R = K[T, T']/(T^2 - a(T')^2 = 4c)$, where $a, c \in K$.

Theorem 5.6

Let $X = \mathrm{Spec}\, A$ be a K-form of $\mathbb{A}^n \times \mathbb{A}^1_*$. Then we have:

(1) There exists a K-subalgebra of A isomorphic to R. Hence there exists a morphism $f : X \to C := \mathrm{Spec}\, R$. The curve C is a K-form of $\mathbb{A}^1_*$.

(2) Each fiber of f is a form of $\mathbb{A}^n$.

(3) If $n = 1, 2$, the morphism f defines an $\mathbb{A}^n$-bundle over C.