$\mathbb{C}^*$ on $\mathbb{C}^3$, with Mariusz

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To

MARIUSZ

passionate mathematician and great friend,

in memory of our long years of joint forays into
AFFINE ALGEBRAIC GEOMETRY wonderland.

I met Mariusz in 1982 when he came to do mathematics in Montreal after the more exiting endeavour of climbing mountains in Alaska. It was the beginning of a long personal friendship, and a long extraordinarily fruitful scientific collaboration. This all started before the advent of e-mail, and so, in a way a bonus, took a lot of travel between Warsaw and Montreal in both directions.
We decided to tackle the linearization of $\mathbb{C}^*$-actions on $\mathbb{C}^3$. Always the optimist, Mariusz gave us half a year to do it. Well, we were actually done 15 years later, relying on our own wits and those of many others, and having to draw on an amazingly extensive panoply of results in what we call now AFFINE ALGEBRAIC GEOMETRY and the rapidly developing theory of OPEN ALGEBRAIC VARIETIES.
So here is the problem.

Notation: Let $T = \mathbb{C}^* = G_m$ (the multiplicative group) and $X = \mathbb{C}^3 = \mathbb{A}^3$ (the affine 3-space).

$\alpha : T \times X \to X$ an effective **action** of the group T on the variety X. Notation: $\alpha(t, p) = t \cdot p$

Algebraically:

k a field (usually $\mathbb{C}$), $k^{[n]} =$ the polynomial algebra in n variables over k.

So $A = k^{[3]} =$ algebra of regular (polynomial) functions on the variety X, $X = Spec(A)$. $\alpha* =$ action of the group T on the algebra A.

Choose variables x, y, z for $A, A = k[x, y, z]$.

The **Action** then is: $t \cdot (x, y, z) = (t \cdot x, t \cdot y, t \cdot z), t \in k^*, k[x, y, z] = k[t \cdot x = f_t(x, y, z), t \cdot y = g_t(x, y, z), t \cdot z = h_t(x, y, z)]$.

Action condition: $t_1 t_2 \cdot (x, y, z) = t_1 \cdot (t_2 \cdot (x, y, z))$, $1 \cdot (x, y, z) = (x, y, z)$.

QUESTION: Can we choose (x, y, z) so that f_t, g_t, h_t are **linear** in x, y, z?

We can then make the action **diagonal**: $t \cdot (x, y, z) = (t^a x, t^b y, t^c z)$,
$a, b, c \in \mathbb{Z} =$ weights of the action.

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Let me return to not necessarily linear actions.

For $\gamma \in \mathbb{Z}, A_\gamma = \{f \in A | t \cdot f = t^\gamma f\}$=set of semi-invariants of weight γ.

Grading: $A = \bigoplus A_\gamma, \gamma \in \mathbb{Z}, A_{\gamma_1} A_{\gamma_2} \subset A_{\gamma_1+\gamma_2}$

T-actions on $X \quad \leftrightarrow \quad \mathbb{Z}$-gradings of A

Definitions: (i) $X^T = \{q \in X | \forall t \in T, t \cdot q = q\}$= **fixpoint set** = $Spec(A/A^\# A), A^\# = \bigoplus A_\gamma, \gamma \neq 0$.
(ii) $A^T = \{f \in A | \forall t \in T, t \cdot f = f\} = A_0$= **subalgebra of invariant functions**.
(iii) $\pi : X \to X//T = Spec(A^T)$, the **categorical quotient**.
$X//T$ is normal and parametrizes the **closed** orbits: Each orbit has a unique closed orbit in its closure. $X^T \subset X//T$ canonically.

The fixpoint theorem(Bialinicky-Birula, Shafarevich): There exists a fixpoint, q say.

Smith theory, Floyd's theorem: W a reasonably nice topological space with a $\mathbb{Z}/p\mathbb{Z}$-action, p a prime. If W has the $\mathbb{Z}/p\mathbb{Z}$-homology of a point, then so does $W^{\mathbb{Z}/p\mathbb{Z}}$.

Local linearization(Koras): A $T = \mathbb{C}^*$-action on $\mathbb{C}^3$ is holomorphically linearizable in a T-invariant open nbhd. of a fixpoint q.

The nature of the fixpoint set: X^T is non-empty, smooth and irreducible.

Let q be a fixpoint,

a, b, c the weights of the induced **diagonal** action on the **tangentspace** $T_q X$.

They are unique up to permutation and replacement of a, b, c by $-a, -b, -c$. With this understanding, the they are independent of the choice of q and called the **weights of the action**.

The action is **effective** if and only if $GCD(a_1, a_2, a_3) = 1$.

Observation: The dimension of X^T is equal to the number of 0-weights.

Definition-Observation: (i) The action is **fixpointed**, i. e., each orbit has a fixpoint in its closure, $\Leftrightarrow X^T = X//T$
iff
(ii) the action is unmixed, i. e., if $A_\gamma \neq 0$, then $A_{-\gamma} = 0 \Leftrightarrow$ each of a, b, c is non-negative (or non-positive).

Theorem on fixpointed actions (Bialinicky-Birula, Kambayashi-R, Bass-Haboush): Y smooth, affine with a fixpointed T-action. Then

$$\pi : Y \to Y//T$$

is a vectorbundle with linear action on the fibers.

Put $\delta = dim(X//T)$, $\quad \tau = dimX^T$. We have the following cases.

- 1. $\delta = 0 = \tau$ $\quad (+, +, +)$, fixpointed,
- 2. $\delta = 1 = \tau$ $\quad (0, +, +)$, fixpointed
- 3. $\delta = 2 = \tau$ $\quad (0, 0, +)$, fixpointed
- 4. $\delta = 2, \tau = 1$ $\quad (-, 0, +)$, semi-hyperbolic case
- 5. $\delta = 2, \tau = 0$ $\quad (-, +, +)$, hyperbolic case

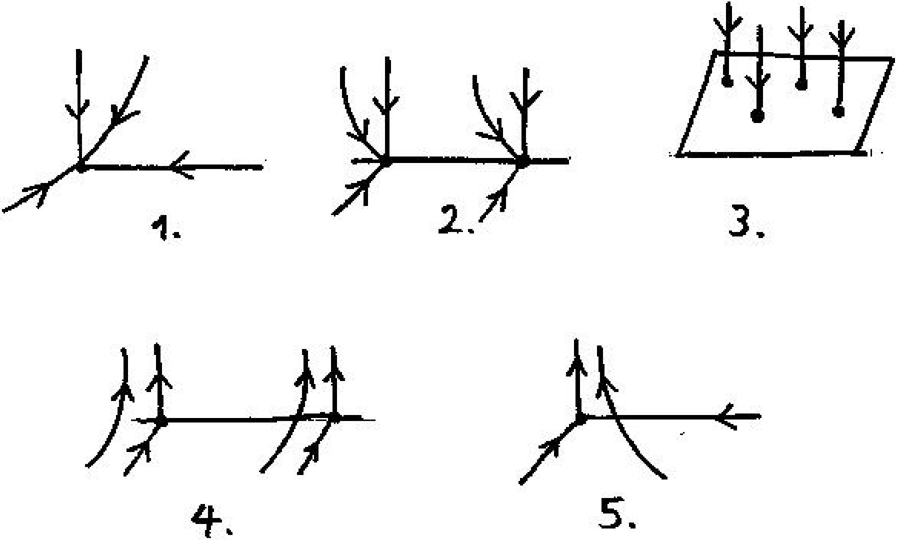

FIGURE 1

Case 1: X is a vectorspace with linear action.

Case 2: X is a vectorbundle with (two-dimensional) fiber E over $X^T = X//T \simeq \mathbb{A}^1 = \mathbb{C}$. So $X \simeq E \times X//T$.

Case 3: X is a vectorbundle with (one-dimensional) fiber E over $Q = X//T$. Q is a retract of an affine space, by **Quillen-Suslin**

$$\mathbb{A}^3 = X \simeq Q \times \mathbb{A}^1.$$

By the **Cancellation Theorem for** $\mathbb{A}^2$ (Fujita, Miyanishi-Sugie): $Q \simeq \mathbb{A}^2$.

Remarks: 1) Here we encounter for the first time the theory of **open surfaces**. The key to the cancellation theorem is that Q has **negative logarithmic Kodaira dimension**.
Mariusz became one of the outstanding experts in this field, with many beautiful results to his credit.

2) It is clear that in higher-dimensional cases of T-actions with just one non-zero weight **linearizability** is equivalent to **cancellation**. There are counter examples to cancellation in positive characteristic in dimension 3 built on the existence of **exotic lines** (Asanuma, Gupta). So we have exotic G_m-actions in dimension 4.

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Case 4 (Koras-R): We have $L = X^T \simeq \mathbb{A}^1$ (Smith theory). The **nullcone** $\pi^{-1}(\pi(X^T))$ is a sort of skeleton of the action, it is the closure of the union of **orbits with a limit point in** X^T. Here it is the union of two invariant hypersurfaces $U = X^+$ and $V = X^-$ on which the action is fixpointed. They are smooth (by local linearizability), therefore planes. Also, $U \cap V = L$. By the **Epimorphism Theorem** (Abhyankar-Moh, Suzuki), L is a coordinate line in both. Linearization follows.

Remarks: 1)We find $X//T \simeq \mathbb{A}^2$, X^T a coordinate line in $X//T$. Alternatively we could have used the
Characterization Theorem(Miyanishi-Sugie): A smooth, factorial affine 2-fold of negative Kodaira dimension is $\mathbb{A}^2$.

2) The argument generalizes to codimension 2 torus actions on $\mathbb{A}^n$ with positive dimensional fixpoint set (Koras-R).

3) In positive characteristic there exist exotic (non-coordinate) lines $f(x, y) = 0$ in the (x, y)-plane, e.g., $f = y^4 + x + x^6$ in characteristic 2. The Weisfeiler 3-fold $W : uv = f(x, y)$ has semi-hyperbolic T-action $t \cdot (x, y, u, v) = (x, y, tu, t^{-1}v)$. W^T is the exotic line $f(x, y) = 0$ in $W//T = Spec(k[x, y])$. It is not known whether $W \simeq A^3$, but $W \times \mathbb{A}^1 \simeq \mathbb{A}^4$ (Asanuma).If yes, the action is not linearizable.

One can make similar examples over $\mathbb{R}$ with knotted lines in $\mathbb{R}^3$ (Shastri) and in the holomorphic category with exotic lines in $\mathbb{C}^2$ (Derksen, Kutzschebauch).

Case 5 This is a long story.

Let q be the unique fixpoint. We write the weights as

$$a_1, a_2, a_3, \quad a_1 < 0, a_2 > 0, a_3 > 0.$$

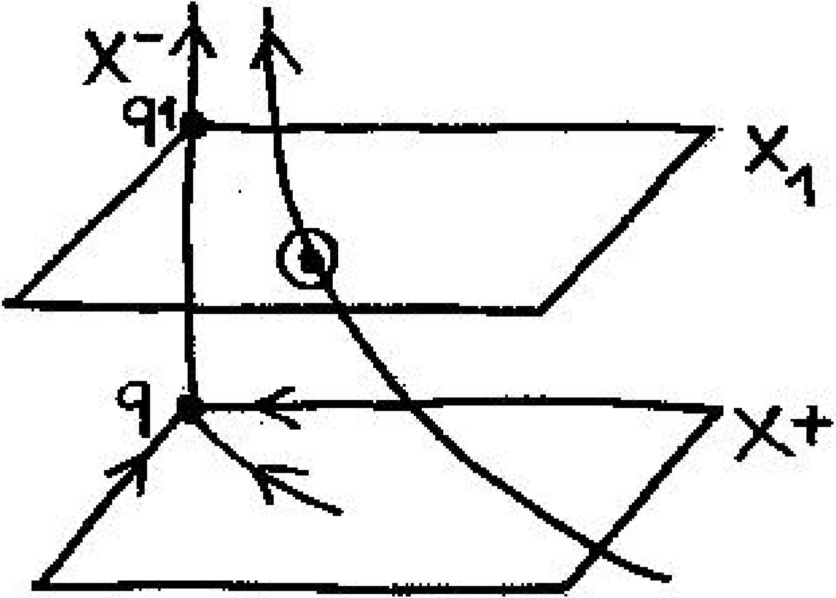
The **null cone** $\pi^{-1}(\pi(q))$ is $\quad X^+ \cup X^-$,

$$X^+ = \{x | lim_{t \to 0} x = q\}, \quad X^- \{x | lim_{t \to \infty} x = q\}.$$

We have $X^+ = F^{-1}(0)$, where F is an irreducible semi-invariant of weight a_1.

$$X^+ \simeq \mathbb{A}^2, \quad X^- \simeq \mathbb{A}^1.$$

The general orbit is a hyperbola going off to infinity along X^- and X^+.


X⁻
q1
X1
q
X+

The a_1-roots of unity $\omega_{\mathbf{a_1}}$ act on the transversal slice

$$X_1 = F^{-1}(1)$$

and restriction of the quotient morphism gives an isomorphism

$$X//T \simeq X_1/\omega_{a_1}.$$

The Big Theorem (Koras-R):

$$X//T \simeq \mathbb{A}^2/\omega_{a_1},$$

where ω_{a_1} acts diagonally with weights

$$a_2, a_3 \quad mod(a_1),$$

i.e., as expected for a linear action.

One can state and prove a version of the Big Theorem in Open Surfaces Theory without reference to quotients.

By standard arguments
1) $X//T$ has negative Kodaira dimension.
2) $X//T$ is contractible. (Kraft-Petrie-Randall, Koras-R)

Theorem(Koras-R): Let S be a contractible affine normal surface with only quotient singular points. If S has negative Kodaira dimension, then so does the smooth locus $S_0 = S \setminus SingS$.

Theorem(Gurjar-Koras-R): Let G be a reductive group acting on $\mathbb{A}^n$ with two-dimensional quotient. Then the quotient is isomorphic to $\mathbb{A}^2/\Omega$, Ω a finite group.

We found it important to learn to **reconstruct X eqivariantly from X_1**.
Lemma(Koras-R): An

$$\omega_{a_1}\text{-equivariant morphism (isomorphism) } \phi : X_1 \to \mathbb{A}^2$$

extends, after suitable modification of ϕ, to a

$$T\text{-equivariant morphism (isomorphism) } \Phi : X \to \mathbb{A}^3,$$

where T acts diagonally with weights a_1, a_2, a_3.
Idea: For $p \in X \setminus X^+$ find $t \in T$ such that $t \cdot p \in X_1$ and put $\Phi(p) = t^{-1} \cdot \phi(t \cdot p)$. Massage ϕ at the $\mathbb{A}^2$-end so that Φ extends to X^+.

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If the **weights are pairwise relatively prime**, then one can see that

$$X_1 \simeq_e \mathbb{C}^2 \text{ (and hence } X \simeq_e \mathbb{C}^3 \text{)}$$

by looking at the **ramification (trivial on the smooth part)** in

$$X_1 \to X//T \simeq X_1/\omega_{a_1} \simeq \mathbb{A}^2/\omega_{a_1} \leftarrow \mathbb{A}^2.$$

For general weights we have no such information on X_1. So we **divide** by pairwise GCD's of the weights.
Reduction of weights: Put $\alpha_1 = GCD(a_2, a_3)$, $\alpha_2 =$... Then

$$a_1 = a_1'\alpha_2\alpha_3, \quad a_2 = a_2'\alpha_1\alpha_3, \quad a_3 = a_3'\alpha_1\alpha_2$$

with

$$-a_1', a_2', a_3' > 0 \text{ and } \textbf{reduced} \text{ (pairwise relatively prime).}$$

Key point: If d is a divisor of an α_i, then ω_d acts with only **one non-zero weight** and hence

X/ω_d is a smooth affine threefold,
X/ω_d is contractible (again by K-P-R or Koras-R),
X^{ω_d} is a T-invariant smooth hypersurface.
T/ω_d acts on X/ω_d, with weights a_i and $a_j/d, j \neq i$. The quotient is unchanged.

The roadblock: **we do not know whether** $X^{\omega_d} \simeq \mathbb{A}^3$. We enlarge the scope of our investigation and consider

the class of T-varieties **LQ**:

(i) $Z = Spec(R)$ a smooth affine threefold with a hyperbolic T-action,
(i) Z is contractible,
(ii) $Z//T \simeq T_q Z//T$.

The above considerations about weights carry over to such Z.

The Z in **LQ** are sometimes called KR-threefods.

The point is,

by the Big Theorem, X (as a T-variety) is in **LQ**.

If Z is in **LQ**, so is Z/ω_d, d as above.

Take note that if $Z \in$ **LQ**, then Z is an **exotic affine space** (diffeomorphic to $\mathbb{R}^6$, Ramanujam, Dimca, Hamm,...)

Strategy: Give an explicit description of **LQ** that allows to recognize the Z that are not $\mathbb{A}^3$ when we disregard the action.

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Strategy: Give an explicit description of **LQ** that allows to recognize the Z that are not $\mathbb{A}^3$ when we disregard the action.

We have an equivariant morphism

$$\rho : Z = Spec(R) \to Z/\omega_{\alpha_3}\omega_{\alpha_2}\omega_{\alpha_1} = W = Spec(B)$$

with

$$W \simeq_e \mathbb{A}^3$$

with reduced weights.

Let me describe the key example.
Let Z be the cubic hypersurface in $\mathbb{C}^4$

$$x + x^2y + u^2 + v^3 = 0.$$

T acts with weights

$$6, -6, 3, 2 \text{ on } x, y, u, v.$$
$$a_1 = -6, a_2 = 3, a_3 = 2 \text{ are the weights of the action on } Z.$$

Note $\alpha_2 = 3, \alpha_3 = 2$. We have

$R = \mathbb{C}[x, y, u, v] = \Gamma(Z)$,
$R^{\omega_2} = \mathbb{C}[x, y, u^2 = \mu, v] = \mathbb{C}[x, y, v]$,
$R^{\omega_3} = \mathbb{C}[x, y, u, v^3 = \nu] = \mathbb{C}[x, y, u]$,
$R^{\omega_6} = \mathbb{C}[x, y, \mu, \nu] = k[x, y, \mu] = k[x, y, \nu] = \Gamma(W), x + x^2y + \mu + \nu = 0.$

μ and ν are homogeneous coordinates on W, but in **different coordinate systems**.

$$\rho : Z \to W$$

is a bi-cyclic cover ramified over the planes $U : \mu = 0$ and $V : \nu = 0$. They intersect in X^- and a closed orbit, and intersect the plane W_1 in two lines L_1, L_2, say, that meet **normally** in the origin and one additional point. μ and ν and Z are determined by the weights and the lines. The reduced weights are $-1, 1, 1$.

Think of L_1, L_2 as a straight line and a parabola, $\mu = 0$ and $x + x^2 + \mu = 0$.

We now argue
1) Z is smooth, acyclic.
2) $\pi_1(W_1 \setminus (L_1 \cup L_2)$ is abelian.
3) $\pi_1(Z) = 1$.
4)Z is contractible.

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Is $Z \simeq \mathbb{C}^3$? Clearly not equivariantly. **Makar-Limanov to the rescue**: x is invariant under every G_a-action on Z. So **No**, R has non-trivial **Makar-Limanov invariant ML**(R)$= \{\xi \in R$ with this property$\} \supsetneq k$

Assume Let $Z \in \mathbf{LQ}$ with $\alpha_2 > 1, \alpha_3 > 1$.

$$\text{Let } Z_1 = Z^+, \quad Z_2 = Z^{\omega_{\alpha_2}}, \quad Z_3 = Z^{\omega_{\alpha_3}}, U_i = \rho(Z_i).$$

Theorem(Koras-R): (i) The $Z_i \quad (U_i)$ with equations $z_i \quad (u_i)$ are smooth homogeneous hypersurfaces of weight $a_i \quad (a_i')$ and

Z is the tri-cyclic covering of $W = \mathbb{A}^3$ ramified to order α_i over U_i.

That is,

$$R = B[z_1, z_2, z_3], \quad \text{with } u_i = z_i^{\alpha_i} \in B.$$

(ii) U_2 and U_3 meet normally in W^+ and $r - 1$ additional closed orbits. $W_1 \cap U_2 \cap U_3$ consists of two lines in W_1 meeting normally in r points.
(iii) There exist homogeneous coordinate systems (u_1, u_2, u_3^*) and (u_1, u_2^*, u_3) for W.
(iv) Write $u_2 = G_2(u_1, u_2^*, u_3)$. Then $A = \mathbb{C}[z_1, z_2, z_2^* = u_2^*, z_3]$ with

$$(*) \quad z_2^{\alpha_2} = G_2(z_1^{\alpha_1}, z_2^*, z_3^{\alpha_3}).$$

as defining equation.
(iv') Similarly for $u_3 = ...$
(v) U_2, U_3 are determined by the lines in (ii).

Remark: The lines can always be realized as a straight L_1 and a line L_2 meeting L_1 in *degree*(L_2) distinct points.

Theorem(Kaliman, Makar-Limanov): If $\epsilon = (r-1)(\alpha_2 - 1)(\alpha_3 - 1) > 0$, then Z has non-trivial Makar-Limanov invariant and $Z \not\cong \mathbb{A}_3$.

Theorem(Koras-R): If $\epsilon = 0$, then $Z \simeq_e \mathbb{A}^3$.

Theorem (Koras-R): The process is reversible, the triple of weights and pair of lines meeting normally in the plane W_1 determine an element of **LQ** as a tri-cyclic cover of W as in the above theorem.

Remark: $\mathbb{Z}^\epsilon = \pi_2(Z \setminus Z^+)$

Proposition(Koras-R): 1) Let Z be in **LQ** with $r \geq 2$. Then the Kodaira dimension is at most 2. It is 2 for large valuues of the α_i.
2) There are non-$\mathbb{A}^3$-examples of Z dominated by $\mathbb{A}^3$, in particular examples with negative Kodaira dimension, e.g., the cubic discussed above.

Theorem(Koras-R): Actions of G_m on $\mathbb{A}^3$ are linearizable over any field of characteristic 0.

Theorem(many contributors): GL_3 is, up to conjugacy, the unique maximal reductive subgroup of $Aut(\mathbb{A}^3)$.

Theorem(Gurjar, Koras, Masuda, Miyanishi, R): A $\mathbb{C}^*$-action on $\mathbb{C}^4$ that fixes a variable is linearizable.

Question: Are finite group actions on $\mathbb{C}^3$ linearizable, e.g., is every involution conjugate to a linear one?

Problem : Start working on co-dimension 2 multiplicative actions on $\mathbb{A}^n$, e.g., $\mathbb{C}^{*2}$ on $\mathbb{C}^4$, with a unique fixpoint. By results mentioned above, quotients are o.k., but there are many cancellation type problems. Mariusz and I talked about this last August.

THE END

except for all the exciting work that remains to be done, unfortunately without Mariusz.