ZARISKI CANCELLATION FOR SURFACES

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(joint with Hubert FLENNER and Shulim KALIMAN)

May 15, 2018

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The biregular version asks as to when

$$X \times \mathbb{A}^n \cong X' \times \mathbb{A}^n \Rightarrow X \cong X'$$

where X, X' are affine algebraic varieties.

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ZCP - FIRST ANSWERS

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"YES" if $X = \mathbb{A}^2$ **and** $\operatorname{char} k = 0$, $\bar{k} = k$
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"YES" if $X = \mathbb{A}^2$ **and** $\operatorname{char} k = 0$, $\bar{k} = k$
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"NO" if $X = \mathbb{A}^3_k$ **and** $\operatorname{char} k = p > 0$
(N. Gupta 2013, using an Asanuma's construction)

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- STRONG ZARISKI FACTOR if any isomorphism $\Phi\colon X \times \mathbb{A}^n \xrightarrow{\cong} X' \times \mathbb{A}^n$ restricts to an isomorphism $\phi\colon X \xrightarrow{\cong} X'$ yielding a commutative diagram

$$\begin{array}{ccc} X \times \mathbb{A}^n & \xrightarrow{\Phi} & X' \times \mathbb{A}^n \\ \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & X' \end{array}$$

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REMARK In higher dimensions, examples were constructed by Dubouloz ('07), Finston and Maubach (2008), Jelonek ('09-'10), Dubouloz, Moser-Jauslin and Poloni ('11), K. Masuda ('17).

DEFINITION A **PARABOLIC $\mathbb{G}_m$-SURFACE** is a normal affine surface with an $\mathbb{A}^1$-fibration $\pi\colon X \to B$ over a smooth affine curve B equipped with an effective $\mathbb{G}_m$-action along the fibers of π.

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- X has no other singular point.

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REMARK For a smooth X the result was also obtained by Adrien Dubouloz ($'16$) with a different proof. A *smooth* parabolic $\mathbb{G}_m$-surface is a line bundle over an affine curve.

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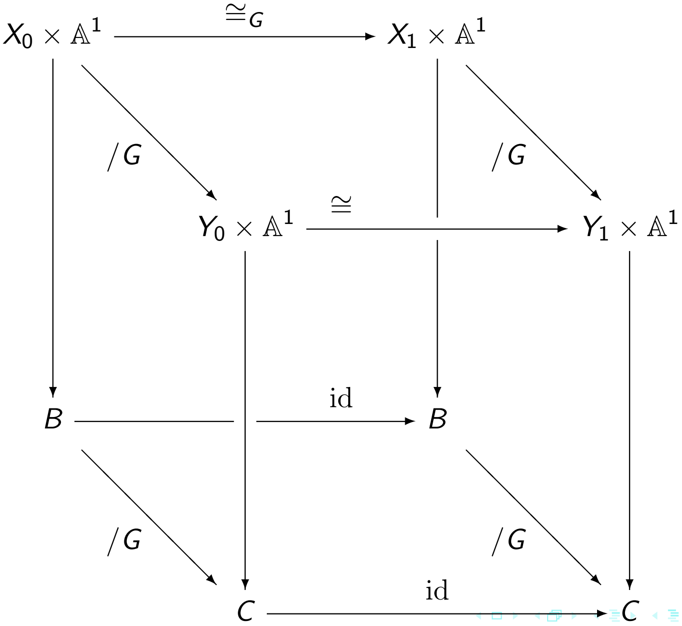
The covering trick allows to reduce the Zariski Cancellation Problem for general affine surfaces $\mathbb{A}^1$-fibered over affine curves to its $\mathbb{Z}/d\mathbb{Z}$ equivariant version for GDF surfaces, due to the following commutative diagram.


$X_0 \times \mathbb{A}^1$
$\cong_G$
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$/G$
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where $\breve{B} \to B$ is an isomorphism outside the points $b_1, \ldots, b_t$ that correspond to degenerate fibers of π, and $\breve{B}$ has n_i points $\{b_{ij}\}$ over b_i if $\pi^{-1}(b_i)$ consists of n_i (reduced) $\mathbb{A}^1$-components.

DEFINITION A divisor D on $\breve{B}$ is a formal sum

$$D = \sum_{j=1}^{N} m_j \breve{b}_j \,,$$

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The Picard group of $\breve{B}$ is the quotient group

$$\mathrm{Pic}(\breve{B}) = \mathrm{Div}(\breve{B})/\mathrm{Princ}(\breve{B}) \,.$$

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$$X \times \mathbb{A}^1 \cong_B X' \times \mathbb{A}^1 \Leftrightarrow L \cong L' .$$

In fact, X is a Zariski factor.

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The **LEVEL** of L_{ij} is the tree distance $l_{ij} = \mathrm{dist}\,(L_{ij}, R_i)$.

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The **TYPE DIVISOR** is

$$\mathrm{tp}(\pi) = -\sum_{i,j} l_{ij} b_{ij} \in \mathrm{Div}(\breve{B})\,.$$

THEOREM *The cylinders over two GDF surfaces* $X \to B$ *and* $X' \to B$ *with the same DF-quotient* $\breve{B}$ *are isomorphic over* B *if and only if their type divisors are linearly equivalent on* $\breve{B}$.

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Figure: The "bush" $\tilde{\Gamma}$ and the "spring bush" $\hat{\Gamma}$ have the same type divisors

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- Cox rings technique.

Cox rings play an important role in the proof of Theorem 1. Next we stay on the two other tools.

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where $\bar{\rho}_i : \bar{X}_i \longrightarrow_B \bar{X}_{i-1}$ is a blowup with a smooth reduced center.

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Given a point $b \in B$ we identify R with the fiber $\{b\} \times \mathbb{P}^1$ of $\mathrm{pr}_1 \colon \bar{X}_0 \to \bar{B}$. Choose n_1 points on R, and let $\bar{\rho}_1 : \bar{X}_1 \longrightarrow_B \bar{X}_0$ be the blowup of these points with exceptional divisor $E_1 = \bar{F}_1 + \ldots + \bar{F}_{n_1}$.

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$$X_1 = \bar{X}_1 \setminus (R' \cup S'_\infty \cup \bar{F}_\infty)$$

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where $S_\infty \subset \bar{X}_0$ is the section at infinity and $\bar{F}_\infty \subset \bar{X}_0$ is the fiber at infinity.
Then $X_1 \to B$ is a GDF surface with a unique reducible fiber $\pi_1^*(b) = F_1 + \ldots + F_{n_1}$.

The center of $\bar{\rho}_2 : \bar{X}_2 \longrightarrow_B \bar{X}_1$ consists of n_2 points on E_1 distributed between the components F_i of $E_1 \setminus S_\infty$ according to the edges of Γ.

A GDF surface with a given fiber tree

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Continuing in this way we construct a GDF surface $X_m \to B$ where m is the height of Γ. This surface has a unique degenerate fiber $\pi_m^{-1}(b)$ whose fiber tree is Γ.

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This leads to a coarse moduli space $\mathcal{M}(B, \mathcal{D})$
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the number of edges of $\mathcal{D}$.

The cylinder $\mathcal{X} = X \times \mathbb{A}^1$ can also be obtained via a sequence of affine modifications

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Each component of the exceptional divisor $\mathcal{E}_i$ of $\tilde{\rho}_i \colon \mathcal{X}_i \to_B \mathcal{X}_{i-1}$ is isomorphic to $\mathbb{A}^2$.

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This leads to the following rigidity result.

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The hard part of the proof consists to establish a bijection between the moduli space of cylinders and the Picard group $\mathrm{Pic}(\check{B})$. To do so we transform any graph divisor to a shruberry without changing the cylinder.

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OPEN QUESTIONS

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QUESTION *Assume that* $\mathrm{Aut}\,(\mathcal{X})$ *is transitive in codimension 2. Is it true that the* $\mathbb{A}^1$*-fibration* $X \to \mathbb{A}^1$ *has at most one reducible fiber?*