The mass of asymptotically hyperbolic manifolds

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Total energy is useful in one-dimensional classical mechanics
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2. But less so for higher dimensional gravitating systems (many body Kepler problem: Xia’s finite-time ejections to infinity).
1 Total energy is useful in one-dimensional classical mechanics

2 But less so for higher dimensional gravitating systems (many body Kepler problem: Xia’s finite-time ejections to infinity)

3 energy and mass are *not always* the same
Mass, momentum, etc., arise as obstructions in gluing problems.
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$m \geq 0$ for AF metrics $\implies$ existence (Schoen 1984, all dim)

for the Yamabe problem
Mass or energy?
What is it good for anyway? some good news in the asymptotically flat case

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2. \( m \geq 0 \) for AF metrics \( \iff \) existence (Schoen 1984, all dim) and compactness (Khuri, Marques, Schoen 2018, dim \( n \leq 24 \), sharp) for the Yamabe problem
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3. $m \geq 0$ for AF metrics $\implies$ suitably regular static black holes are Schwarzschild in all dimensions
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3. $m \geq 0$ for AF metrics $\implies$ suitably regular static black holes are Schwarzschild in all dimensions

4. Hollands and Wald (2016): variational identities involving total mass for AF metrics can be used to prove existence of instabilities in “black strings”
Spacetime variational methods: “Noether charge” à la Wald (~ 1990) ≡ geometric Hamiltonian methods à la Kijowski-Tulczyjew (1979)

2. Hamiltonians for asymptotic symmetries:

\[ H(\partial_t, \{ t = 0 \}) \] is the energy

If \( g \) suitably approaches a background \( g \) with a Killing vector field \( X \), then the Hamiltonian is

\[ H(X, S) := \frac{1}{2} \int_{\partial S} \left( U_{\nu\lambda} - U_{\nu\lambda} \right) |g = g| dS_{\nu\lambda}, \tag{1} \]

\[ U_{\nu\lambda} = \frac{1}{8} \pi \sqrt{|\det g|} g_\alpha[^{\nu\delta}]_\beta \nabla_\alpha X_\beta, \tag{2} \]

\[ U_{\nu\lambda}^\beta = \frac{1}{16} \pi \sqrt{|\det g|} 2 g^{\beta\gamma} \nabla_\kappa \left( e^2 g^{\gamma[^{\lambda} g^{\nu}\kappa]} \right), \tag{3} \]

where \( \nabla \) is the covariant derivative of \( g^{\mu\nu} \) and \( e^2 = \det g \det g \). \( \tag{4} \)
How to define mass
Spacetime methods

1 Spacetime variational methods: “Noether charge” à la Wald (~ 1990) ≡ geometric Hamiltonian methods à la Kijowski-Tulczyjew (1979): “\(H(\partial_t, \{t = 0\})\)” is the energy

2 Hamiltonians for asymptotic symmetries: If \(g\) suitably approaches a background \(\bar{g}\) with a Killing vector field \(X\), then the Hamiltonian is

\[
H(X, \mathcal{I}) := \frac{1}{2} \int_{\partial \mathcal{I}} (U^{\nu\lambda} - U^{\nu\lambda}|_{g=\bar{g}}) dS_{\nu\lambda},
\]

where

\[
U^{\nu\lambda} = U^{\nu\lambda}_\beta X^\beta - \frac{1}{8\pi} \sqrt{|\det g|} g^{\alpha[\nu} \delta^{\lambda]} \nabla_\alpha X^\beta,
\]

\[
U^{\nu\lambda}_\beta = \frac{2|\det \bar{g}|}{16\pi \sqrt{|\det g|}} g_{\beta\gamma} \nabla_\kappa (e^2 g^{\gamma[\lambda} g^{\nu]\kappa}),
\]

where \(\nabla\) is the covariant derivative of \(\bar{g}_{\mu\nu}\) and

\[
e^2 \equiv \frac{\det g}{\det \bar{g}}.
\]
Asymptotically locally hyperbolic (ALH) metrics

Asymptotically hyperbolic if \((N^{n-1}, \hat{h})\) is the unit round sphere

g = \ell^2 x^{-2} \left( dx^2 + (1 - \frac{k}{4} x^2)^2 \hat{h} + x^n \mu \right) + o(x^{n-2}) dx^i dx^j,

\hat{h} = \hat{h}_{AB}(x^C) dx^A dx^B, \quad \mu = \mu_{AB}(x^C) dx^A dx^B,

\ell > 0 \text{ is a constant related to } \Lambda, \quad \hat{h} \text{ is a Riemannian metric on } N^{n-1} \text{ with scalar curvature}

R[\hat{h}] = (n - 1)(n - 2)k, \quad k \in \{0, \pm 1\}.

The mass aspect function is

\[ \theta := \text{tr}_\hat{h} \mu \]

uniquely defined unless the conformal infinity is a round sphere

The total mass is

\[ m_0 = c_n \int_{N^{n-1}} \theta, \quad m_i = c_n \int_{S^{n-1}} \theta x^i \]

(defines a “Minkowskian” vector on a sphere)
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Hyperbolic mass (also known as *holographic energy*, cf. “*holographic stress-energy tensor*”).

- We only have satisfactory understanding of mass and related invariants in the asymptotically **Euclidean** setting. (Spectacular progress by Schoen and Yau 2017.)

- Asymptotically **hyperbolic** setting: Positivity? Spin structure or other topological restrictions? Sharp and insightful inequalities in higher dim? e.g., on spin manifolds with spherical infinity, in

\[ E^2 \geq |\vec{j}|^2, \quad (6) \]

where $\vec{j}$ is the total angular momentum.
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  \[ E^2 \geq |\vec{j}|^2, \]  
  \[ (6) \]
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Hyperbolic mass (also known as *holographic energy*, cf. “holographic stress-energy tensor”).

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- Asymptotically *hyperbolic* setting:Positivity? Spin structure or other topological restrictions? Sharp and insightful inequalities in higher dim? e.g., on spin manifolds with spherical infinity, in three space-dimensions

\[ E^2 - |\vec{\rho}|^2 \geq -\Lambda/3 \left( |\vec{c}|^2 + |\vec{j}|^2 + 2 |\vec{c} \times \vec{j}| \right), \quad (6) \]

where $\vec{j}$ is the total angular momentum and $\vec{c}$ the centre of mass.
What backgrounds $g$?

- For simplicity, assume vacuum Einstein equations throughout:

$$R(g)_{\mu\nu} = c_n \Lambda g_{\mu\nu}$$  \hspace{1cm} (7)
What backgrounds $g$?  
What are the spacelike manifolds $\mathcal{I}$ we are interested in?

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- This talk: mostly $\Lambda < 0$
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- For simplicity, assume vacuum Einstein equations throughout:
  \[ R(g)_{\mu\nu} = c_n \Lambda g_{\mu\nu} \quad (7) \]
- This talk: mostly $\Lambda < 0$
- What kind of spacelike hypersurfaces are compatible with (7) when $\Lambda \neq 0$
Constraint equations, cosmological constant $\Lambda$

Does the curvature scalar know about $\Lambda$? ($\rho = j^k = 0$ in vacuum)

• The scalar constraint equation:

$$R(g) = 16\pi \rho + |K|^2 - (\text{tr}K)^2 + 2\Lambda$$  \hspace{1cm} (8)

where $\rho$ is the energy density of matter fields, $R(g)$ is the scalar curvature of the space metric
constraint equations, cosmological constant \( \Lambda \)

Does the curvature scalar know about \( \Lambda \)? (\( \rho = j^k = 0 \) in vacuum)

- The scalar constraint equation:

\[
R(g) = 16\pi \rho \pm |K|^2 - (\text{tr}K)^2 + 2\Lambda \\
= 16\pi \rho + |\hat{K}|^2 - \left(\frac{n-1}{n}\right)(\text{tr}K)^2 + 2\Lambda,
\]

where \( \rho \) is the energy density of matter fields, \( R(g) \) is the scalar curvature of the space metric, and \( \hat{K} \) is the trace-free part of the extrinsic curvature tensor \( K \).
Constraint equations, cosmological constant $\Lambda$

Does the curvature scalar know about $\Lambda$? ($\rho = j^k = 0$ in vacuum) assume tr$K$ to be constant

- The scalar constraint equation:

$$R(g) = 16\pi\rho + |K|^2 - (\text{tr}K)^2 + 2\Lambda$$

$$= 16\pi\rho + |\hat{K}|^2 - \frac{(n-1)}{n}(\text{tr}K)^2 + 2\Lambda$$

$$=: 2\tilde{\Lambda}$$

where $\rho$ is the energy density of matter fields, $R(g)$ is the scalar curvature of the space metric, and $\hat{K}$ is the trace-free part of the extrinsic curvature tensor $K$. 

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The mass of asymptotically hyperbolic manifolds
Constraint equations, cosmological constant $\Lambda$

Does the curvature scalar know about $\Lambda$? ($\rho = j^k = 0$ in vacuum) assume $\text{tr} K$ to be constant

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where $\rho$ is the energy density of matter fields, $R(g)$ is the scalar curvature of the space metric, and $\hat{K}$ is the trace-free part of the extrinsic curvature tensor $K$.

- You can fool around with $\Lambda$ by playing with the trace of $K$

$$K \rightarrow K + ag \quad \Rightarrow \quad \tilde{\Lambda} \rightarrow \tilde{\Lambda} - \frac{(n - 1)}{2n}(2a\text{tr}K + a^2)$$
Constraint equations, cosmological constant $\Lambda$

Does the curvature scalar know about $\Lambda$? ($\rho = j^k = 0$ in vacuum) assume $\text{tr} K$ to be constant

- The scalar constraint equation:

$$R(g) = 16\pi\rho + |K|^2 - (\text{tr} K)^2 + 2\Lambda$$

$$= 16\pi\rho + |\hat{K}|^2 - \frac{(n-1)}{n}(\text{tr} K)^2 + 2\Lambda,$$

where $\rho$ is the energy density of matter fields, $R(g)$ is the scalar curvature of the space metric, and $\hat{K}$ is the trace-free part of the extrinsic curvature tensor $K$.

- You can fool around with $\Lambda$ by playing with the trace of $K$

$$K \rightarrow K + ag \quad \Rightarrow \quad \tilde{\Lambda} \rightarrow \tilde{\Lambda} - \frac{(n-1)}{2n}(2a\text{tr} K + a^2)$$

- This is compatible with the vector constraint equation:

$$D_i(K_{ik} - \text{tr} K g^{ik}) = 8\pi j^k$$
• **Corollary**: The Trautman-Bondi mass $m_{TB}$ is the same as the hyperbolic mass
• **Corollary**: The Trautman-Bondi mass $m_{TB}$ is the same as related to the hyperbolic mass (⚠️ pure trace $K$ + constraint equations + $\Lambda = 0 \implies$ no gravitational radiation ⚠️)
• **Corollary:** The Trautman-Bondi mass $m_{TB}$ is the same as related to the hyperbolic mass (pure trace $K$ + constraint equations + $\Lambda = 0 \implies$ no gravitational radiation)

• **Corollary:** positivity theorems for asymptotically hyperbolic initial data ($\Lambda < 0$) translate to angular momentum bounds with $\Lambda = 0$

$$m_{TB} \geq \frac{|\text{tr}K|}{3} |\vec{J}|, \quad m_{TB} \geq \frac{|\text{tr}K|}{3} |\vec{c}|,$$

where $\vec{J}$ is the total angular momentum and $\vec{c}$ the centre of mass.
• **Corollary**: The Trautman-Bondi mass $m_{TB}$ is the same as related to the hyperbolic mass (pure trace $K$ + constraint equations + $\Lambda = 0 \implies$ no gravitational radiation)

• **Corollary**: positivity theorems for asymptotically hyperbolic initial data ($\Lambda < 0$) translate to angular momentum bounds with $\Lambda = 0$ on CMC hypersurfaces $\mathcal{I}$ when there is no-radiation at the conformal boundary of $\mathcal{I}$

$$m_{TB} \geq \frac{|\text{tr}K|}{3} \, |\vec{J}|, \quad m_{TB} \geq \frac{|\text{tr}K|}{3} \, |\vec{c}|,$$

where $\vec{J}$ is the total angular momentum and $\vec{c}$ the centre of mass.
Asymptotically Anti-de Sitter metrics

- Asymptotically anti-de Sitter metrics:

\[ g \to r \to \infty \bar{g} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2, \quad V = r^2 + 1. \]
• Asymptotically anti-de Sitter metrics:
\[ g \rightarrow r \rightarrow \infty \quad \bar{g} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2, \quad V = r^2 + 1. \]

• Elementary positive energy theorem: in a suitable gauge, for 
\[ h := g - \bar{g} \text{ small, } (E := H(\partial_t, \{ t = 0 \})) \]

\[ E \geq \int_M \left[ R - \bar{R} + \frac{n-2}{16n} |Dh|_g^2 \right] V. \]
Asymptotically Anti-de Sitter metrics

- Asymptotically anti-de Sitter metrics:
  \[
  g \rightarrow r \rightarrow \infty \quad \bar{g} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2, \quad V = r^2 + 1.
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- Elementary positive energy theorem: in a suitable gauge, for \( h := g - \bar{g} \) small, \( (E := H(\partial_t, \{ t = 0 \})) \)
  \[
  E \geq \int_M \left[ R - \bar{R} + \frac{n - 2 - \epsilon}{8n} |\bar{D} tr h|^2_g + \frac{1 - \epsilon}{4} |\bar{D} \hat{h}|^2_g \right. \\
  - \left. \frac{1 + \epsilon}{1} |\hat{h}|^2_g \right] V \sqrt{\det \bar{g}} \\
  \geq \int_M \left[ R - \bar{R} + \frac{n - 2}{16n} |\bar{D} h|^2_g \right] V.
  \]
• Asymptotically anti-de Sitter metrics:

\[ g \rightarrow_{r \to \infty} \overline{g} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2, \quad V = r^2 + 1. \]

• Elementary positive energy theorem: in a suitable gauge, for \( h := g - \overline{g} \) small, \((E := H(\partial_t, \{t = 0\}))\)

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E \geq \int_M \left[ R - \overline{R} + \frac{n - 2 - \epsilon}{8n} |\overline{Dtr} h|^2_{\overline{g}} + \frac{1 - \epsilon}{4} |\overline{Dh}|^2_{\overline{g}} - \frac{1 + \epsilon}{2} |\hat{h}|^2_{\overline{g}} \right] V \sqrt{\det \overline{g}} \]

\[
\geq \int_M \left[ R - \overline{R} + \frac{n - 2}{16n} |Dh|^2_{g} \right] V.
\]

• but no stability: arbitrarily small generic perturbations of initial data for the spherically symmetric Einstein-scalar field equations produce arbitrarily small black holes (?).
Asymptotically Anti-de Sitter metrics

Geometric formulae for total energy (Ashtekar Romano 1992; Herzlich 2015; PTC, Barzegar, Hörzinger 2017), space-dimension \( n \)

\[
\mathbf{g} \rightarrow_{r \rightarrow \infty} \bar{\mathbf{g}} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2, \quad V = r^2 + 1.
\]

• For any Killing vector \( X \) of \( \bar{\mathbf{g}} \) we have

\[
H_b (X, \mathcal{L}) = \frac{1}{16(n - 2)\pi} \lim_{R \to \infty} \int_{t=0, r=R} X^\nu Z^\xi W^\alpha{}_{\beta}{}^{\nu \xi} dS_{\alpha \beta},
\]

where \( W^\alpha{}_{\beta}{}^{\nu \xi} \) is the Weyl tensor of \( \mathbf{g} \) and \( Z = r \partial_r \) is the dilation vector field.
Asymptotically Anti-de Sitter metrics

Geometric formulae for total energy (Ashtekar Romano 1992; Herzlich 2015; PTC, Barzegar, Höerzinger 2017), space-dimension $n$

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where $W^{\alpha \beta}_{\nu \xi}$ is the Weyl tensor of $g$ and $Z = r \partial_r$ is the dilation vector field

- Riemannian version, asymptotically hyperbolic Riemannian metrics $g$, $R^i_j$ is the Ricci tensor of $g$:

\[
H_b(X, \mathcal{L}) = -\frac{1}{16(n-2)\pi} \lim_{R \rightarrow \infty} \int_{r=R} X^0 V Z^j (R^i_j - \frac{R}{n} \delta^i_j) dS_i.
\]
Asymptotically Anti-de Sitter metrics
Komar-type formula (PTC, Barzegar, Hörzinger 2017), space-dimension $n$

\[
g \to_{r \to \infty} \bar{g} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2, \quad V = r^2 + 1.
\]

- If $X$ is a Killing vector of both $g$ and $\bar{g}$ we have

\[
H_b(X, \mathcal{I}) = \lim_{R \to \infty} \left\{ \frac{n-1}{16(n-2)\pi} \int_{r=R} X^{[\alpha;\beta]} dS_{\alpha\beta} - \frac{\Lambda}{4(n-2)(n-1)n\pi} \int_{r=R} X^{\alpha} Z^{\beta} dS_{\alpha\beta} \right\},
\]

where $\Lambda < 0$ is the cosmological constant.
Other asymptotic backgrounds: Kottler-Birmingham metrics
Static vacuum solutions of Einstein equations with a negative cosmological constant

\[ g_m = -V_m^2 dt^2 + V_m^{-2} dr^2 + r^2 h_\kappa, \quad V_m = r^2 + \kappa - \frac{2m}{r^{n-2}}. \]

where \( h_\kappa \) is a \( t \)- and \( r \)-independent Einstein metric on a \((n - 1)\)-dim compact manifold, with scalar curvature \( R(h) = (n - 1)(n - 2)\kappa \).

Question: Is (9) an absolute lower bound for vacuum black holes? Yes for solutions with a constant negative mass aspect function.
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Static vacuum solutions of Einstein equations with a negative cosmological constant

\[ g_m = -V_m^2 dt^2 + V_m^{-2} dr^2 + r^2 h_\kappa, \quad V_m = r^2 + \kappa - \frac{2m}{r^{n-2}}. \]

where \( h_\kappa \) is a \( t \)- and \( r \)-independent Einstein metric on a \( (n - 1) \)-dim compact manifold, with scalar curvature \( R(h) = (n - 1)(n - 2)\kappa \).

- The mass of \( g_m \) relative to \( \bar{g} := g_0 \) is proportional to \( m \)
Other asymptotic backgrounds: Kottler-Birmingham metrics

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- The manifolds are singular unless the \( V_m \)'s have positive zeros, which then correspond to **black hole horizons**

\[ (9) \]
Other asymptotic backgrounds: Kottler-Birmingham metrics
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\[ m \geq -\frac{(n-1)(n-3)/2}{(n+1)(n-1)/2}. \]
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- Question: Is (9) an absolute lower bound for vacuum black holes?
Other asymptotic backgrounds: Kottler-Birmingham metrics

Lee & Neves, n = 3, 2015
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- Question: Is (9) an absolute lower bound for vacuum black holes? yes for solutions with a constant negative mass aspect function

Piotr T. Chruściel
The mass of asymptotically hyperbolic manifolds
$g_m = \pm V_m^2 dt^2 \theta^2 + V_m^{-2} dr^2 + r^2 (dr^2 - dt^2 + h_0')$, $V_m = \frac{2m}{r^{n-2}}$.

where $h_0'$ is a $t$, $\theta$, and $r$-independent Ricci flat metric on a $(n-3)$-dim compact manifold.

- Naked singularity for $m < 0$. 
\[ g_m = V_m^2 d\theta^2 + V_m^{-2} dr^2 + r^2 ( -dt^2 + h'_0 ), \quad V_m = r^2 \left( -\frac{2m}{r^{n-2}} \right). \]

where \( h'_0 \) is a \( t-, \theta-, \text{and } r\)-independent Ricci flat metric on a \((n-3)\)-dim compact manifold.

- Naked singularity for \( m < 0 \).
- Complete cusp at infinity when \( m = 0 \).
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- For $m > 0$ the zero-sets of $V_m$ are smooth totally-geodesic submanifolds ("core geodesics" in $n = 3$) when the period of $\theta$ is appropriately chosen, depending upon $m$. 
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- For \( m > 0 \) the zero-sets of \( V_m \) are smooth totally-geodesic submanifolds ("core geodesics" in \( n = 3 \)) when the period of \( \theta \) is appropriately chosen, depending upon \( m \).
- The mass relative to \( g_0 \) can be arbitrarily negative, proportional to the negative of \( m \).
Horowitz-Myers Instantons
Woolgar's version of the Horowitz-Myers conjecture

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- The mass relative to \( g_0 \) can be arbitrarily negative, proportional to the negative of \( m \).
- Conjecture: these are local minima of energy.
Horowitz-Myers Instantons
the Woolgar-Horowitz-Myers conjecture for nearby metrics

\[ h = g - \bar{g}, \hat{h} = \text{trace-free part of } h: \]

\[
m = \int_M \left[ (R - \bar{R}) V + \left( \frac{n + 2}{8n} |D \phi|^2 + \frac{1}{4} |D \hat{h}|^2 \right. \right.
\]

\[
- \frac{1}{2} \hat{h}^{i\ell} \hat{h}^{jm} \bar{R}_{\ell m ij} - \frac{n + 2}{2n} \phi \hat{h}^{ij} \bar{R}_{ij} + \frac{n(n^2 - 4)}{8n^2} \phi^2
\]

\[
- \frac{1}{2} \left( |\bar{\psi}|^2 - \bar{\psi}^i D_i \phi \right) V + \left( h^k_i \bar{\psi}^i + \frac{1}{2} \phi \bar{\psi}^k \right) D_k V
\]

\[
+ (O \left( |h|^3 \right) + O \left( |h| |Dh|^2 \right)) V
\]

\[
+ O \left( |h|^2 |Dh| |DV| \right) \sqrt{\det g}. \tag{10}
\]


$h = g - \bar{g}$, $\hat{h} =$ trace-free part of $h$:

$$m = \int_M \left[ (R - \bar{R}) V + \left( \frac{n + 2}{8n} |D\phi|^2_g + \frac{1}{4} |D\hat{h}|^2_g ight) 
- \frac{1}{2} \hat{h}^{ie} \hat{h}^{jm} R_{\ell m ij} - \frac{n + 2}{2n} \phi \hat{h}^{ij} \bar{R}_{ij} + \frac{n(n^2 - 4)}{8n^2} \phi^2 \right) V + \left( 
+ O\left( |h|^3_g \right) + O\left( |h|_g |\bar{D}h|^2_g \right) \right) V 
+ O\left( |h|^2_g |\bar{D}h|_g \right) |\bar{D}V|_g \right] \sqrt{\det g}.$$  

(10)

gauge terms
Horowitz-Myers Instantons
the Woolgar-Horowitz-Myers conjecture for nearby metrics

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\]
\[
\left. \right) V \right] \sqrt{\det g}. \quad (10)
\]

\textit{gauge/errror terms} ???

Piotr T. Chruściel
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A sharper Poincaré inequality?

gauge terms error terms ?? Sharper Poincaré inequality?
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\]
\[
\left. \right) V \right] \sqrt{\det g}. \quad (10)
\]

\text{gauge/terms error/terms ??? Sharper Poincaré inequality? Incidentally: } \text{Uniqueness theorems} \text{ for the Horowitz-Myers instanton by Galloway and Woolgar, and by M. Anderson} \]
Reminder: Asymptotically locally hyperbolic (ALH) metrics

Asymptotically hyperbolic if \((N^{n-1}, \hat{h})\) is the unit round sphere

\[
g = \ell^2 x^{-2} \left( dx^2 + (1 - \frac{k}{4} x^2)^2 \hat{h} + x^n \mu \right) + o(x^{n-2}) dx^i dx^j,
\]

\[
\hat{h} = \hat{h}_{AB}(x^C) dx^A dx^B, \quad \mu = \mu_{AB}(x^C) dx^A dx^B,
\]

\(\ell > 0\) is a constant related to \(\Lambda\), \(\hat{h}\) is a Riemannian metric on \(N^{n-1}\) with scalar curvature

\[
R[\hat{h}] = (n - 1)(n - 2) k, \quad k \in \{0, \pm 1\}.
\]  \(\text{(11)}\)

The mass aspect function is

\[
\theta := \text{tr}_{\hat{h}} \mu
\]

uniquely defined unless the conformal infinity is a round sphere

The total mass is

\[
m_0 = c_n \int_{N^{n-1}} \theta, \quad m_i = c_n \int_{S^{n-1}} \theta x^i
\]

(defines a “Minkowskian” vector on a sphere).
Theorem

Let \((M^n, g)\), \(4 \leq n \leq 7\), be a \(C^{n+5}\)-conformally compactifiable asymptotically locally hyperbolic (ALH) Riemannian manifold diffeomorphic to \([r_0, \infty) \times N^{n-1}\) with a compact boundary \(N_0 := \{r_0\} \times N^{n-1}\) and with well defined total mass. Suppose that:

1. The mean curvature of \(N_0\) satisfies \(H < n - 1\), where \(H\) is the divergence \(D_i n^i\) of the unit normal \(n^i\) pointing into \(M\).
2. The scalar curvature \(R = R[g]\) of \(M\) satisfies \(R \geq -n(n - 1)\).
3. Either \((N, \hat{h})\) is a flat torus, or \((N, \hat{h})\) is a nontrivial quotient of a round sphere.

Then the mass of \((M^n, g)\) is nonnegative, \(m \geq 0\).
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Theorem

Let \((M^n, g)\) be an ALH manifold, \(n \geq 4\). For all \(\epsilon > 0\) there exists a metric \(g_\epsilon\) which coincides with \(g\) outside of an \(\epsilon\)-neighborhood of the conformal boundary at infinity, satisfies \(R[g_\epsilon] \geq R[g]\), such that

1. \(g_\epsilon\) has a pure monopole-dipole mass aspect function \(\Theta_\epsilon\) if \((N^{n-1}, \hat{h})\) is conformal to the standard sphere, and has constant mass aspect function otherwise;
2. the associated energy-momentum satisfies

\[
\lim_{\epsilon \to 0} m_0^c = m_0, \quad m_i^c = m_i, \quad \text{if } (N^{n-1}, \hat{h}) \text{ round } S^{n-1};
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(12)
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