Decidability questions for Cuntz-Krieger algebras and their underlying dynamics

Søren Eilers
eilers@math.ku.dk

Department of Mathematical Sciences
University of Copenhagen

IMPAN, January 17, 2018

Content

1. Shifts of finite type
2. C^*-algebras
3. Systematic approach
4. Questions

Outline

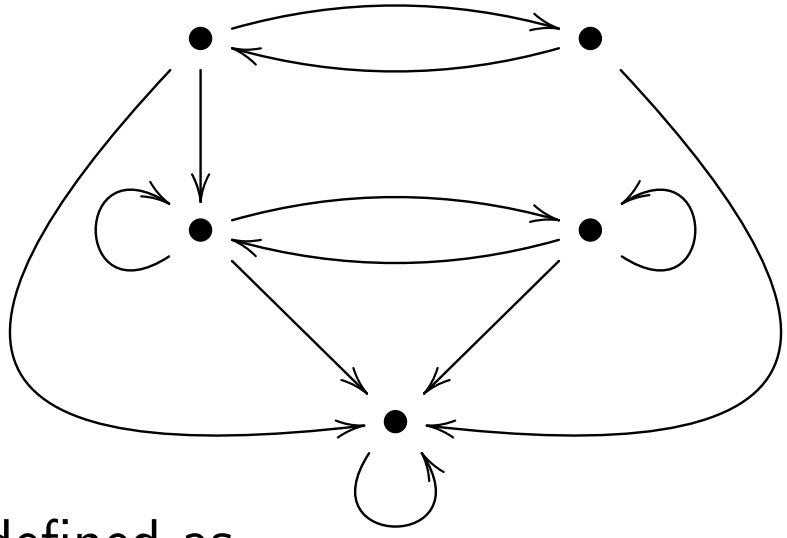
1 Shifts of finite type

2 C^*-algebras

3 Systematic approach

4 Questions

To a finite graph $E = (E_0, E_1, r, s)$ such as



we associate X_E defined as

$$\mathsf{X}_E = \{(e_n) \in (E_0)^{\mathbb{Z}} \mid r(e_n) = s(e_{n+1})\}$$

Note that X_E is closed in the topology of $(E_0)^{\mathbb{Z}}$ and comes equipped with a shift map $\sigma : \mathsf{X}_E \to \mathsf{X}_E$ which is a homeomorphism. We call X_E a **shift space** (2-sided, of finite type) over the **alphabet** E_0.

Definition

The **suspension flow** SX of a shift space X is $X \times \mathbb{R}/\sim$ with

$$(x,t) \sim (\sigma(x), t-1)$$

Note that SX has a canonical $\mathbb{R}$-action.

Definitions

Let X and Y be shift spaces.

- X is conjugate to Y (written $X \simeq Y$) if there is a shift-invariant homeomorphism $\varphi : X \to Y$.
- X is flow equivalent to Y (written $X \sim_{\mathrm{FE}} Y$) if there is an orientation-preserving homeomorphism $\psi : SX \to SY$

Question

Are these notions decidable for shifts of finite type?

Question

Are these notions decidable for shifts of finite type?

Theorem (Boyle-Steinberg)

Flow equivalence is decidable among shifts of finite type.

Definition

Let $A \in M_n(\mathbb{Z}_+)$ and $B \in M_m(\mathbb{Z}_+)$ be given. We say that A is **elementary equivalent** to B if there exist $D \in M_{n\times m}(\mathbb{Z}_+)$ and $E \in M_{m\times n}(\mathbb{Z}_+)$ so that

$$A = DE \qquad B = ED.$$

The smallest equivalence relation on $\bigcup_{n\geq 1} M_n(\mathbb{Z}_+)$ is called **strong shift equivalence**.

Let E_A be the graph with adjacency matrix A. We abbreviate $\mathsf{X}_A = \mathsf{X}_{E_A}$.

Theorem (Williams)

$\mathsf{X}_A \simeq \mathsf{X}_B$ *if and only if* A *is strong shift equivalent to* B.

Definition

We say that that A and B are **shift equivalent** of lag ℓ when there exist $D \in M_{n\times m}(\mathbb{Z}_+)$ and $E \in M_{m\times n}(\mathbb{Z}_+)$ so that

$$A^\ell = DE \qquad B^\ell = ED \qquad AD = DB \qquad EA = BE.$$

Strong shift equivalence implies shift equivalence.

Theorem (Kim-Roush)

Shift equivalence is decidable.

It took decades to disprove

William's conjecture

Shift equivalence coincides with strong shift equivalence.

and indeed it is a prominent open question if conjugacy is decidable for shifts of finite type.

Outline

1 Shifts of finite type

2 C^*-algebras

3 Systematic approach

4 Questions

For any A, let $\mathcal{G}_A$ denote the dimension group

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots$$

and let s_A denote the induced map

$$\begin{array}{ccccccc} \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \cdots \\ & \searrow & & \searrow & & & \\ \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \cdots \end{array}$$

Let $(\mathfrak{A}_A, \varphi_A)$ denote the corresponding stable AF algebra and $*$-automorphism.

Theorem (Krieger)

The following are equivalent

1. A *are* B *are shift equivalent;*
2. $(\mathcal{G}_A, \mathsf{s}_A) \simeq (\mathcal{G}_B, \mathsf{s}_B)$*;*
3. *There is a* $*$*-isomorphism* $\psi : \mathfrak{A}_A \to \mathfrak{A}_B$ *and unitaries* $u_n \in M(\mathfrak{A}_B)$ *so that*

$$u_n(\varphi_B \circ \psi)u_n^* \to \psi \circ \varphi_A$$

Singular and regular vertices

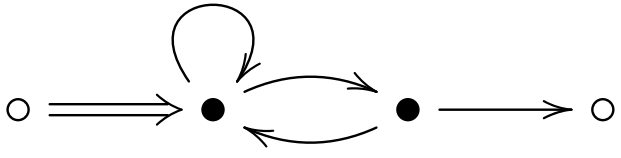
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$

Definition

v is *singular* if v is a sink or an infinite emitter. v is *regular* if it is not singular.



Graph algebras

Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital precisely when E has finitely many vertices.

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = zs_e$$

induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Definition

$$\mathfrak{D}_E = \overline{\mathrm{span}}\{s_\alpha s_\alpha^* \mid \alpha \text{ path of } E\}$$

Note that $\mathfrak{D}_E$ is commutative and that

$$\mathfrak{D}_E \subseteq \mathfrak{F}_E = \{a \in C^*(E) \mid \forall z \in \mathbb{T} : \gamma_z(a) = a\}$$

$\mathfrak{D}_E$ has spectrum X_A when $E = E_A$ arises from an essential and finite matrix A. This fundamental case was studied by Cuntz and Krieger, using the notation $\mathcal{O}_A = C^*(E_A)$. In fact $\mathfrak{F}_{E_A} \otimes \mathbb{K} \simeq \mathfrak{A}_A$.

Cuntz-Krieger correct "if"

4.1. **Theorem.** *If T_1 and T_2 are flow equivalent then $(\bar{\mathcal{O}}_{T_1}, \bar{\mathcal{D}}) \sim (\bar{\mathcal{O}}_{T_2}, \bar{\mathcal{D}})$.*

Theorem (Carlsen-E-Ortega-Restorff, Matsumoto-Matui)

$(\mathcal{O}_A \otimes \mathbb{K}, \mathfrak{D}_A \otimes c_0) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \mathfrak{D}_B \otimes c_0) \iff \mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$

Cuntz-Krieger correct "if"

2.17. **Proposition.** *If σ_A and σ_B are topologically conjugate, then there is an isomorphism of $\mathcal{O}_A$ onto $\mathcal{O}_B$ transforming $\mathcal{D}_A$ into $\mathcal{D}_B$, $\Phi_{A|\mathcal{D}_A}$ into $\Phi_{B|\mathcal{D}_B}$ and $(\lambda_t^A)_{t\in\mathbb{T}}$ into $(\lambda_t^B)_{t\in\mathbb{T}}$.*

Theorem (Brix-Carlsen)

The following are equivalent

1. $\mathsf{X}_A^+ \simeq \mathsf{X}_B^+$
2. $(\mathcal{O}_A, \gamma_z^A, \mathfrak{D}_A, \Phi_A|\mathfrak{D}_A) \simeq (\mathcal{O}_B, \gamma_z^B, \mathfrak{D}_B, \Phi_B|\mathfrak{D}_B)$

Not predicted by Cuntz-Krieger

Theorem (Carlsen-Rout, Matsumoto)

$$(\mathcal{O}_A \otimes \mathbb{K}, \mathfrak{D} \otimes c_0, \gamma \otimes \mathrm{Id}) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \mathfrak{D} \otimes c_0, \gamma \otimes \mathrm{Id})$$
$$\Longleftrightarrow$$
$$\mathsf{X}_A \simeq \mathsf{X}_B$$

Theorem (E-Restorff-Ruiz-Sørensen,Boyle-Steinberg)

$*$-isomorphism and stable $*$-isomorphism of Cuntz-Krieger algebras are classifiable by K-theory and further decidable.

Not predicted by Cuntz-Krieger

Theorem (Carlsen-Rout, Matsumoto)

$$(\mathcal{O}_A \otimes \mathbb{K}, \mathfrak{D} \otimes c_0, \gamma \otimes \mathrm{Id}) \simeq (\mathcal{O}_B \otimes \mathbb{K}, \mathfrak{D} \otimes c_0, \gamma \otimes \mathrm{Id})$$
$$\Longleftrightarrow$$
$$\mathsf{X}_A \simeq \mathsf{X}_B$$

Theorem (E-Restorff-Ruiz-Sørensen, Boyle-Steinberg)

$*$-isomorphism and stable $*$-isomorphism of Cuntz-Krieger algebras are classifiable by K-theory and further decidable.

Outline

1 Shifts of finite type

2 C^*-algebras

3 Systematic approach

4 Questions

Definition

With x, y, z $\in \{0, 1\}$ we write

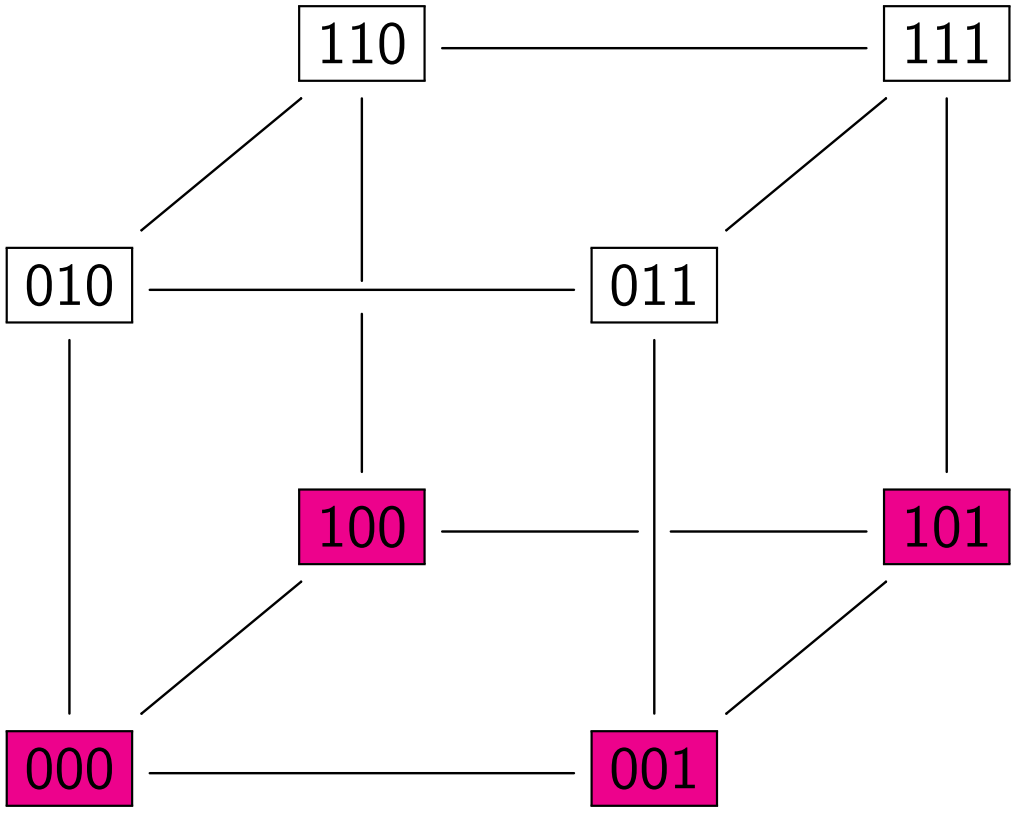
$$E \xrightarrow{\text{xyz}} F$$

when there exists a $*$-isomorphism $\varphi : C^*(E) \otimes \mathbb{K} \to C^*(F) \otimes \mathbb{K}$ which additionally satisfies

- $\varphi(1_{C^*(E)} \otimes e_{11}) = 1_{C^*(F)} \otimes e_{11}$ when x $= 1$
- $\varphi \circ (\gamma \otimes \mathrm{Id}) = (\gamma \otimes \mathrm{Id}) \circ \varphi$ when y $= 1$
- $\varphi(\mathfrak{D}_E \otimes c_0) = \mathfrak{D}_F \otimes c_0$ when z $= 1$.

We use "111+" to indicate the type of isomorphism considered in the Brix-Carlsen paper (requires absense of infinite emitters).

Unital graph algebras



Theorem (E-Restorff-Ruiz-Sørensen, Boyle-Steinberg)

$E \overset{\text{x0z}}{\longrightarrow} F$ *is decidable.*

Theorem (Carlsen-E-Ortega-Restorff, Matsumoto-Matui)

$E_A \xrightarrow{001} E_B \iff \mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$

Theorem (Carlsen-Rout, Matsumoto)

$E_A \xrightarrow{011} E_B \iff \mathsf{X}_A \simeq \mathsf{X}_B$

Theorem (Bratteli-Kishimoto)

$E_A \xrightarrow{010} E_B \iff \mathsf{X}_A$ *is shift equivalent to* X_B *when* $\mathsf{X}_A, \mathsf{X}_B$ *are mixing.*

Theorem (Carlsen-E-Ortega-Restorff, Matsumoto-Matui)

$E_A \xrightarrow{001} E_B \iff \mathsf{X}_A \sim_{\mathrm{FE}} \mathsf{X}_B$

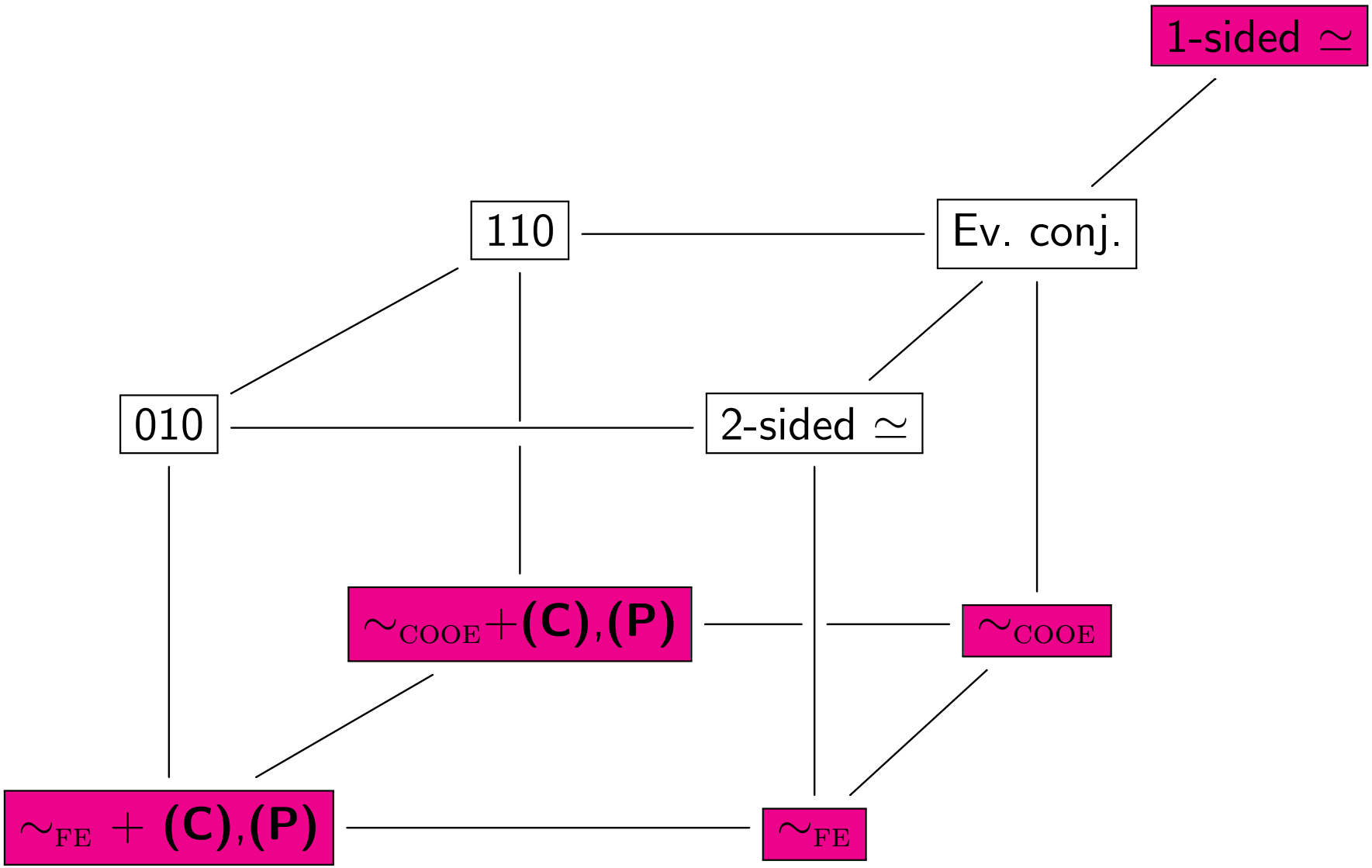
Theorem (Carlsen-Rout, Matsumoto)

$E_A \xrightarrow{011} E_B \iff \mathsf{X}_A \simeq \mathsf{X}_B$

Theorem (Bratteli-Kishimoto)

$E_A \xrightarrow{010} E_B \iff \mathsf{X}_A$ *is shift equivalent to* X_B *when* $\mathsf{X}_A, \mathsf{X}_B$ *are mixing.*

Cuntz-Krieger algebras/shift spaces

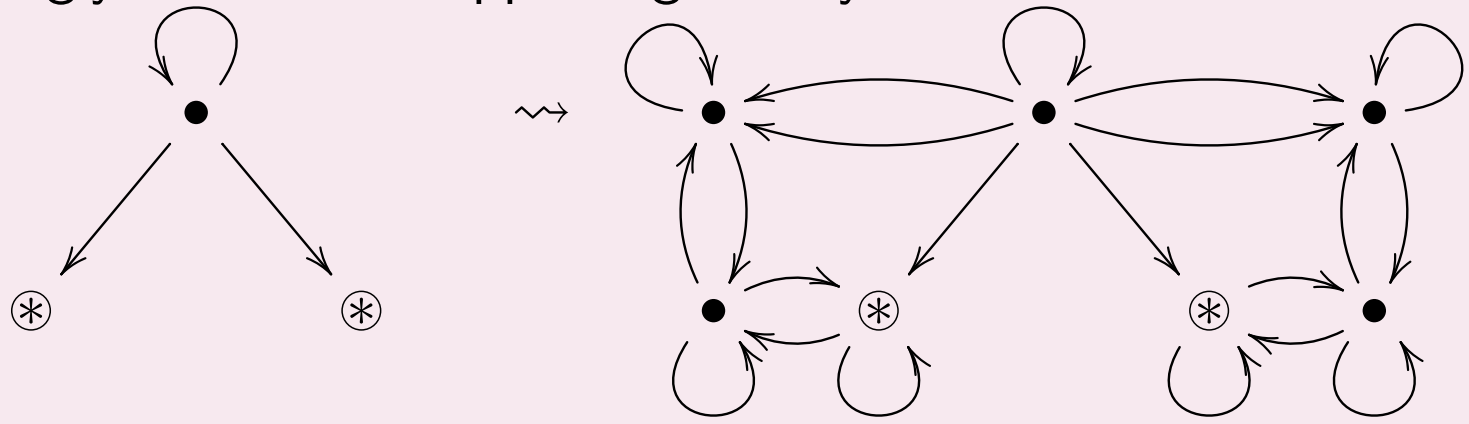


Move **(C)**

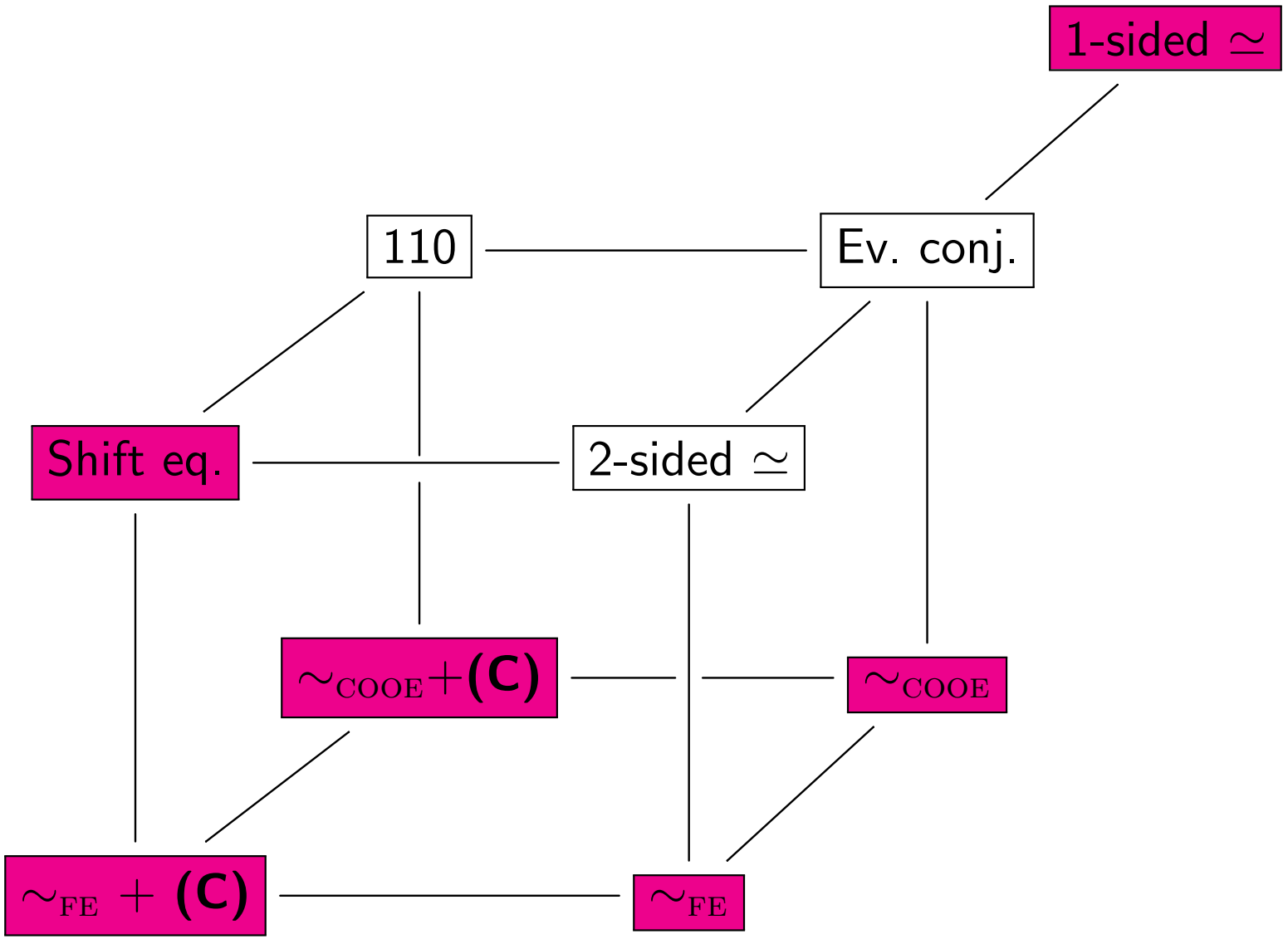
"Cuntz splice" on a vertex supporting two cycles

Move **(P)**

"Butterfly move" on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles



Cuntz-Krieger algebras for **mixing** shift spaces



Outline

1 Shifts of finite type

2 C^*-algebras

3 Systematic approach

4 Questions

Independence

Suppose $\mathsf{xyz} \geq \mathsf{x'y'z'}$, then of course

$$E \xrightarrow{\mathsf{xyz}} F \implies E \xrightarrow{\mathsf{x'y'z'}} F$$

Counterexamples show that no other relations of this type hold, except possibly

$$E \xrightarrow{\mathsf{x10}} F \implies E \xrightarrow{\mathsf{x01}} F$$

In fact when A, B are mixing, we have

$$E_A \xrightarrow{010} E_B \implies E_A \xrightarrow{001} E_B$$

Key questions

- Does $E_A \xrightarrow{010} E_B$ encode shift equivalence in general? If so, are there versions of shift equivalence for general unital graph algebras?
- Is 010-equivalence decidable in general?
- Does shift equivalence imply flow equivalence? If so, are there versions of this for general unital graph algebras?
- How do the invariants establishing failure of the Williams conjecture manifest in C^*-algebras?
- Can all these kinds of isomorphisms be described by moves?