OPTIMAL FRAMEWORK STRUCTURES AND LINKS TO OPTIMAL TRANSPORT PROBLEM

KAROL BOŁBOTOWSKI

The central physical context of this talk will be the engineering problem of finding a planar framework that, by means of tension and compression, optimally transfers the loading given by a vector-valued measure. Since the beginning of the 20th century it has been established that, even in the case when the support of the loading is finite, an optimal structure being a truss comprising finite number of bars does not exist.

The first problem put forward will be that of finding an optimal infinite truss – a framework consisting of infinite number of straight bars. The sought structure will be modelled as a signed measure on the Cartesian product of planes; the measure will be constrained by the equilibrium with the loading. The form of the minimized integral allows to recognize the problem as a vector version of the broadly investigated Kantorovich optimal transport problem. The existence of solution for the latter problem is well-established, while the optimal infinite truss problem in general fails to have a minimizer. After A.G.M. Michell, this can be remedied by extending the range of candidate structures to tensor-valued measures on the plane that represent stress fields in a hybrid, continuum-discrete framework that is widely called a Michell structure – numerous analytical solutions show that it may comprise curved bars that has been missing in a problem of infinite truss.

The talk will be concluded by readdressing analogical problems in the case of load transfer by bending. The question of existence of solution in the problem of optimal infinite grillage, i.e. a counterpart of a truss, remains open.

ABSTRACT

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AN OPTIMAL TRANSPORT APPROACH TO THE BV LEAST GRADIENT PROBLEM IN 2D

SAMER DWEIK

In this talk, we want to study the $W^{1,p}$ regularity of the solution, in 2D, of the BV least gradient problem

$$\min\left\{\int_\Omega |\nabla u|\,\mathrm{d}x \,:\, u\in BV(\Omega),\, u_{|\partial\Omega}=g\right\}, \tag{0.1}$$

when Ω is a uniformly convex domain, $u_{|\partial\Omega}$ denotes the trace of u and $g:\partial\Omega\mapsto\mathbb{R}$ is a L^1 given function. First, we recall the connection between (0.1) and the Beckmann problem [1] (see [6])

$$\min\left\{\int_\Omega |w|\,\mathrm{d}x \,:\, w\in L^1(\Omega,\mathbb{R}^2),\ \nabla\cdot w=0 \ \text{ in } \mathring{\Omega},\ w\cdot n=f \ \text{ on } \partial\Omega\right\}, \tag{0.2}$$

where f is the tangential derivative of g (i.e. $f=\partial g/\partial\tau$, τ is the tangent on the boundary) and n is the outward normal to $\partial\Omega$. More precisely, in [6], the authors prove that if u is a solution of (0.1), then $w=R_{\frac{\pi}{2}}\nabla u$ solves (0.2), where $R_{\frac{\pi}{2}}$ denotes the rotation with angle $\frac{\pi}{2}$ around the origin. In addition, it is well-known that this problem (0.2) is equivalent to the Monge-Kantorovich one

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma :\ \gamma\in\Pi(f^+,f^-)\right\}, \tag{0.3}$$

where

$$\Pi(f^+,f^-):=\left\{\gamma\in\mathcal{M}^+(\Omega\times\Omega):\ (\Pi_x)_\#\gamma=f^+\ ,\ (\Pi_y)_\#\gamma=f^-\ \right\}$$

and $f=f^+-f^-$.

In fact, one can prove that if γ is a minimizer of (0.3), then the vector measure w_γ defined as follows

$$< w_\gamma,\xi >:=\int_{\Omega\times\Omega}\mathrm{d}\gamma(x,y)\int_0^1\xi((1-t)x+ty)|x-y|\,\mathrm{d}t,\ \text{ for all } \xi\in C(\Omega,\mathbb{R}^2)$$

solves (0.2). And, it is well-known that any minimizer w of (0.2) comes from a minimizer γ of (0.3) (see, for instance, [8]). Moreover, we are able to show, in the case of interest for us, that this minimizer is unique as soon as f^+ or f^- is atomless and Ω is uniformly convex (see [5]).

Then, studying the $W^{1,p}$ regularity of u becomes studying the L^p summability of the optimal flow w. Notice that [2, 3, 4, 7] analyze the L^p summability of the optimal flow w: in dimension 2, for $p < 2$, $w \in L^p(\Omega, \mathbb{R}^2)$ as soon as at least one between the two measures f^+ or f^- is in $L^p(\Omega)$ and, for $p \geq 2$, this requires that both f^+ and f^- will be in $L^p(\Omega)$. However, the L^p summability of w in the case where we transport a measure f^+, concentrated on the boundary, to another one f^-, concentrated also on the boundary, is not known in the literature. In particular, if $f^\pm \in L^p(\partial\Omega)$, is it true that the optimal flow w between f^+ and f^- is in $L^p(\Omega)$?

Using an approximation with atomic measures, we prove that if $f^\pm \in L^p(\partial\Omega)$, then the optimal flow w is in $L^p(\Omega)$, as soon as $p \leq 2$ and Ω is uniformly convex. Consequently, under these assumptions, if the boundary datum g is in $W^{1,p}(\partial\Omega)$, then the solution u is in $W^{1,p}(\Omega)$. Finally, by a counter-example, we show that this result is no longer true if $p > 2$.

References

[1] M. Beckmann, A continuous model of transportation, *Econometrica*, 20, 643-660, 1952.

[2] L. De Pascale, L. C. Evans and A. Pratelli, Integral estimates for transport densities, *Bull. of the London Math. Soc.*, 36, n. 3,pp. 383-395, 2004.

[3] L. De Pascale and A. Pratelli, Regularity properties for Monge Transport Density and for Solutions of some Shape Optimization Problem, *Calc. Var. Par. Diff. Eq*, 14, n. 3, pp.249-274, 2002.

[4] L. De Pascale and A. Pratelli, Sharp summability for Monge Transport density via Interpolation, *ESAIM Control Optim. Calc. Var.*, 10, n. 4, pp. 549-552, 2004.

[5] S. Dweik and F. Santambrogio, L^p bounds for boundary-to-boundary transport densities, and $W^{1,p}$ bounds for the BV least gradient problem in 2D, *cvgmt/paper/3792*, 2018.

[6] W. Górny, P. Rybka and A. Sabra, Special cases of the planar least gradient problem, *Nonlinear Analysis*, 151, pp. 66 – 95, 2017.

[7] F. Santambrogio, Absolute continuity and summability of transport densities: simpler proofs and new estimates, *Calc. Var. Par. Diff. Eq.*, (2009) 36: 343-354.

[8] F. Santambrogio, *Optimal Transport for Applied Mathematicians*, in *Progress in Nonlinear Differential Equations and Their Applications* 87, Birkhäuser Basel (2015).

ANISOTROPIC YOUNG'S MODULUS DESIGN

GRZEGORZ DZIERŻANOWSKI (WITH T. LEWIŃSKI)

ABSTRACT. Free Material Design (FMD) is a method for optimal design of load-carrying structures; it consists in controlling arbitrary properties of 4th order tensor field endowed with certain symmetries. Within this scope, we focus on optimizing the Young modulus, i.e. stress-to-elongation ratio, of material fibers directed along the trajectories of eigenvalues of 2nd order symmetric tensor field representing stresses induced in the material by loads imposed on a structure.

The optimization task under study takes a form of the following constrained variational problem: Determine two variables for which the stress energy functional is minimal: (i) design variable – an anisotropic Young modulus field restricted by a global integral constraint, and (ii) state variable – a divergence-free stress field solving the boundary value problem of linearized elasticity. It turns out that thus posed optimization task reduces to minimizing the functional of linear growth in stresses; this feature is shared by all FMD problems which has yet been developed.

Date: May 7, 2018.

Harnack inequality for degenerate quasilinear elliptic equations

Maria Stella Fanciullo

We prove Harnack inequality and regularity for non negative solutions of a quasilinear degenerate elliptic equations, assuming the coefficients in the structure conditions to belong to suitable Stummel-Kato classes.

STABLE DETERMINATION OF A LAMÉ COEFFICIENT BY ONE INTERNAL MEASUREMENT OF DISPLACEMENT

GIUSEPPE DI FAZIO

The shear modulus μ of an isotropic elastic body is stably recovered by the knowledge of one single displacement field whose boundary data can be assigned independently on the unknown elasticity tensor.

Date: May 11, 2018.

HÖLDER REGULARITY OF ANISOTROPIC LEAST GRADIENT FUNCTIONS

WOJCIECH GÓRNY

Consider the anisotropic least gradient problem in two dimensions

$$\min\{\int_\Omega \phi(Du), \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f\}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary, $f \in C(\partial\Omega)$, and ϕ is an anisotropic norm on $\mathbb{R}^2$. As in the classical least gradient problem, existence and uniqueness of minimizers depend on the geometry on Ω. From here, there are two different scenarios:

(1) The unit ball $B_\phi(0,1)$ is strictly convex. In this case, we have existence and uniqueness of minimizers of the anisotropic least gradient problem for every boundary data $f \in C(\partial\Omega)$ if Ω is strictly convex. Furthermore, if Ω satisfies some form of uniform convexity, then we may obtain regularity estimates regardless of the regularity of ϕ; in particular, if $\partial\Omega \in C^2$ and the curvature is bounded from below, then $f \in C^{0,\alpha}(\partial\Omega)$ implies $u \in C^{0,\alpha/2}(\Omega)$ (the same as in the isotropic case).

(2) The unit ball $B_\phi(0,1)$ is not strictly convex. In this case existence of minimizers is obtained (for uniformly convex domains) using the regularity estimates from case (1). This gives us one minimizer with the same regularity estimates as in the previous paragraph. However, uniqueness of minimizers fails and the resulting minimizers may fail to be $W^{1,1}(\Omega)$ or $SBV(\Omega)$ even for smooth boundary data.

ON THE EXISTENCE OF DICHROMATIC SINGLE ELEMENT LENSES

CRISTIAN E. GUTIÉRREZ

Due to dispersion, light with different wavelengths, or colors, is refracted at different angles. So when white light is refracted by a single lens, in general, each color comes to a focus at a different distance from the objective. This is called chromatic aberration and plays a vital role in lens design. A way to correct chromatic aberration is to build lenses that are an arrangement of various single lenses made of different materials. Our purpose in this talk is to show when is mathematically possible to design a lens made of a single homogeneous material so that it refracts light superposition of two colors into a desired fixed final direction. Two problems are considered: one is when light emanates in a parallel beam and the other is when light emanates from a point source. The mathematical tools used to solve these problems include fixed point theorems and functional differential equations. This is joint work with A. Sabra.

Reference: https://arxiv.org/abs/1801.07223

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SINGULAR SOLUTIONS IN OPTIMAL TRANSPORTATION

YASH JHAVERI

In the optimal transport problem, it is well-known that the geometry of the target domain plays a crucial role in the regularity of the optimal transport. In the quadratic cost case, Caffarelli showed that having a convex target domain is essential in guaranteeing the optimal transport's continuity. In this talk, we shall explore how, quantitatively, important convexity is in producing continuous optimal transports.

OPTIMAL TRANSPORT WITH NON-LOCAL INTERACTION

ARAM KARAKHANYAN

We will consider an optimal transport problem with free boundary such that the transport potential has non-local character.

ON REGULARITY AND STABILITY OF FREE DISCONTINUITIES IN OPTIMAL TRANSPORT

JUN KITAGAWA

Abstract. Regularity of solutions in the optimal transport problem require very rigid geometric hypotheses such as the convexity of certain sets. In more general cases one can consider the question of partial regularity, i.e. in-depth analysis of the structure of singular sets. In this talk I will discuss the finer structure of the set of "free singularities" which arise in an optimal transport problem from a connected set to a disconnected set, along with the stability of such sets under suitable perturbations of the data involved. These results arise from a non-smooth implicit function theorem for convex functions, which is itself of independent interest. This talk is based on joint work with Robert McCann.

Weak solutions of complex Monge-Ampère and m-Hessian equations

Sławomir Kołodziej

Abstract

I will survey some recent developments in the theory of complex Monge-Ampère and m-Hessian equations and their applications in Kähler and Hermitian geometry.

ON THE ISOTROPIC AND CUBIC MATERIAL DESIGN METHODS AND THEIR APPLICATIONS IN 3D PRINTING

TOMASZ LEWIŃSKI

The paper (joint work with S. Czarnecki, R. Czubacki, T. Łukasiak, P. Wawruch) deals with the selected methods belonging to the class of free material design constructed by assuming: a) cubic symmetry (cubic material design, CMD), b) isotropy with: (b1) independent bulk and shear moduli (isotropic material design, IMD), and (b2) fixed Poisson's ratio (Young's modulus design, YMD). In the latter case the Young modulus is the only design variable. The moduli are viewed as non-negative, thus allowing for the appearance of void domains. The paper shows that all these methods (CMD, IMD, YMD) reduce to two mutually dual problems:

(P) the stress-based minimization problem in which the integrand is equal to a norm of the test stress field. The norm $||.||$ involved reflects the type of the constraints imposed;

(P*) the displacement-based problem in which the virtual work is subject to maximization over the adjoint displacement fields associated with strains, the dual norm $||.||^*$ of which is bounded almost everywhere.

Upon solving problem (P) and finding the minimizer one can determine the optimal moduli; they assume non-zero values within the material domain and they vanish outside this domain, thus allowing for cutting out the final shape from the initial design domain.

The YMD method has been made suitable for the 3D printing. The procedure leading from theory to the practice of 3D printing will be shown in detail.

Date: May 7, 2018.

Total variation flow of curves

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Abstract

Let $(\mathcal{N}, g)$ be a complete, n-dimensional Riemannian submanifold in Euclidean space $\mathbb{R}^N$ and let I be a bounded interval. We introduce a natural notion of solution to the formal L^2-gradient flow of the total variation functional

$$TV_g(\boldsymbol{u}) = \int_I |\boldsymbol{u}_x|_g$$

for $\boldsymbol{u} \in H^1(0,T; L^2(I,\mathcal{N})) \cap L^\infty(0,T; BV(I,\mathcal{N}))$. Given any $\boldsymbol{u}_0 \in BV(I,\mathcal{N})$ whose jumps are not too large, we sketch the proof of existence of a solution for arbitrarily large $T > 0$.

An important ingredient of the proof is a *completely local* estimate

$$\int_A |\boldsymbol{u}_x(t,\cdot)|_g \leq \int_A |\boldsymbol{u}_{0,x}|_g$$

for any Borel $A \subset I$. This estimate seems to be new even in the case $\mathcal{N} = \mathbb{R}^n$.

The talk is based on a recent, unpublished joint work with L. Giacomelli.

Kurdyka-Łojasiewicz-Simon inequality for gradient flows in metric spaces

José M. Mazón

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Abstract:

The classical Łojasiewicz inequality and its extensions by Simon and Kurdyka have been a considerable impact on the analysis of the large time behaviour of gradient flow in Hilbert spaces. Our aim is to adapt the classical Kurdyka-Łojasiewicz and Łojasiewicz-Simon inequalities to the general framework gradient flow in metric spaces. We show that the validity of a Kurdyka-Łojasiewicz inequality imply trend to equilibrium in the metric sense, and the Kurdyka-Łojasiewicz inequality has the advantage to derive decay estimates of the trend to equilibrium and finite time of extinction. Also we study the relation between Kurdyka-Łojasiewicz inequality and the existence of talweg. The entropy method have proved to be very useful to study the large time behaviour of solutions to many EDP's. This method is based in the entropy-entropy production/disipation (EEP) inequality, which correspond to Kurdyka-Łojasiewicz inequality, and also in the entropy transportation (ET) inequality. We show that for geodesically convex functionals Kurdyka-Łojasiewicz inequality and entropy transportation (ET) inequality are equivalent. We apply our general results to gradient flow in Banach spaces and in spaces of probability measures with Wasserstein distances. For the energy functional associated with a doubly-nonlinear equations on $\mathbb{R}^N$ we obtain the equivalence between Łojasiewicz-Simon inequality, generalized log-Sobolev inequality and p-Talagrand inequality; also we get decay estimates for its solutions. Finally we apply our results to metric spaces with Ricci curvature bounds from below, getting that, in this context, a p-Talagrand inequality is equivalent to a Łojasiewicz-Simon inequality.

Joint work with Daniel Hauer (Sydney University)

The constrained total variation flow.

Salvador Moll

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Let $(\mathcal{N}, g)$ be a complete, n-dimensional Riemannian manifold (embedded as a surface in Euclidean space $(\mathbb{R}^N, |\cdot|)$ with some $N \geq n$). Given an open, bounded domain $\Omega \subset \mathbb{R}^m$ we consider the formal steepest descent flow with respect to L^2 distance of the functional

$$\boldsymbol{u} \in C^1(\overline{\Omega}, \mathcal{N}) \mapsto TV(\boldsymbol{u}) = \int_\Omega |\nabla \boldsymbol{u}|.$$

The flow is formally given by the system

$$\begin{cases} \boldsymbol{u}_t = \pi_{\boldsymbol{u}} \mathrm{div} \frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} & \text{in }]0,T[\times\Omega, \\ \boldsymbol{\nu}^\Omega \cdot \frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} = 0 & \text{in }]0,T[\times\partial\Omega. \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases}$$

where, $\pi_{\boldsymbol{p}} \colon T_{\boldsymbol{p}}\mathbb{R}^N \equiv \mathbb{R}^N \to T_{\boldsymbol{p}}\mathcal{N}$ denotes the orthogonal projection onto $T_{\boldsymbol{p}}\mathcal{N}$ at a point $\boldsymbol{p} \in \mathcal{N}$.

In this talk I will present some recent results in collaboration with L. Giacomelli, M. Łasica and J. Mazón (see the references below) about existence of solutions in different scenarios:

(a) The case of $\mathcal{N}$ being a semicircle or a hyperoctant of the unit sphere with initial data in $BV(\Omega;\mathcal{N})$.

(b) The general case. Here, I will present existence, uniqueness and asymptotic behavior of solutions in the case of regular initial data.

References

[1] L. Giacomelli, J. Mazón, S. Moll, The 1-Harmonic Flow with Values into $\mathbb{S}^1$, SIAM J. Math. Anal. **45**(3), 1723–1740 (2013).

[2] L. Giacomelli, J. Mazón, S. Moll, The 1-harmonic flow with values in a hyperoctant of the N-sphere, Analysis and PDE, **7**(3), 627–671 (2014)

[3] L. Giacomelli, M. Łasica, S. Moll, Regular 1−harmonic flow, preprint: https://arxiv.org/abs/1711.07460.

Anisotropic TV flow in 2D case.

Piotr Bogusław Mucha

University of Warsaw

I plan to talk about solvability of solutions to the anisotropic two space dimensional total variation flow. The anisotropy defines minimal sets as squares, the dissipation if just in two main direction $x1$ and $x2$. The main language of analysis of this problem is built over determination of evolutions of step type function, more precisely of functions piecewise constant on rectangles (PCR). Since this set is preserved under the considered flow, one can determine completely the evolutions in this class. Particularly, i plan to discuss the case of breaking of facets.

The talk is based on analysis of result of join paper with Salva Moll (Valencia) and Michał Łasica (Rome).

RIGIDITY ISSUES IN CONICAL REGIONS

GIULIO TRALLI

Abstract. We consider partially overdetermined problems in conical domains and constant mean curvature hypersurfaces with boundary suitably attached to a smooth cone. For the case of convex cones, we present respectively a Serrin-type and an Aleksandrov-type result. We will focus on the role of the convexity of the cone and of the gluing between the relative boundary of the domain and the cone. We will also show a rigidity result for constant mean curvature surfaces in starshaped sectors related to possibly non-convex cones. This is a joint work with F. Pacella.

Date: May 6, 2018.

A MODEL FLOW FOR SUBMANIFOLDS WITH CONSTANT CURVATURE

GLEN WHEELER

One of the most basic pursuits in geometry is the understanding of shapes with least bending. In this talk, we interpret bending not as pure curvature but as a derivative of curvature (although linguistically it sounds odd, this is called the jerk), and take an energetic approach toward the analysis of shapes with least jerk. We propose a broad problem in the calculus of variations, on submanifolds with parallel mean curvature vector. As a first step, we study the problem in the geometrically mostly uninteresting case of curves in the plane. Here the gradient flow nevertheless challenges us to come up with new methods. First, we determine the set of equilibria – circles – through an elementary analysis of the Euler-Lagrange equation. Then, we define a scale-invariant energy and study the flow for small enough initial energy. After some effort, we prove convergence in this energetic neighbourhood of the flow to a round circle. Apart from energy estimates, the Lojasiewicz-Simon gradient inequality makes an appearance. We quite carefully establish the gradient inequality in our setting, which although simple, still requires some effort. This is joint work with Ben Andrews (ANU), James McCoy (UoN) and Valentina-Mira Wheeler (UOW).

Date: May 11, 2018.

On concavity of the principal's profit maximization facing agents who respond nonliearly to prices

Kelvin Shuangjian Zhang

Abstract

A monopolist wishes to maximize her profits by finding an optimal price policy. After she announces a menu of products and prices, each agent will choose to buy that product which maximizes his own utility, if positive. The principal's profits are the sum of the net earnings produced by each product sold. These are determined by the costs of production and the distribution of products sold, which in turn are based on the distribution of anonymous agents and the choices they make in response to the principal's price menu. In this talk, we describe a necessary and sufficient condition for the convexity or concavity of the principal's problem, assuming each agent's disutility is a strictly increasing but not necessarily affine (i.e. quasilinear) function of the price paid. Concavity when present, makes the problem more amenable to computational and theoretical analysis; it is key to obtaining uniqueness and stability results for the principal's strategy in particular. Even in the quasilinear case, our analysis goes beyond previous work by addressing convexity as well as concavity, by establishing conditions which are not only sufficient but necessary, and by requiring fewer hypotheses on the agents' preferences. This talk represents joint work with my supervisor Robert McCann.