Pointwise products of functions in Hardy spaces and their duals

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It is known that the product of two functions, one in $H^1(\mathbb{R}^n)$ and the other one in $\mathrm{BMO}\,(\mathbb{R}^n)$, can be given a meaning and may be written as the sum of an integrable function and a function in $H^{\log}(\mathbb{R}^n)$, which is defined as the space of functions f such that the non tangential maximal function $\mathcal{M}f$ satisfies

$$\int \frac{\mathcal{M}f(x)}{\log(e+|x|)+\log(e+\mathcal{M}f(x))}dx < \infty.$$

That is, f belongs to a Musielak-Orlicz Hardy space, a notion that generalizes weighted Hardy Orlicz spaces since, in some sense, the weight depends on the point.

It was natural to see what is the situation for the Hardy space $H^p(\mathbb{R}^n)$ when $p < 1$. For $p \in (0,1)$ and $\frac{\beta}{n} = 1/p - 1$, let $\dot{\Lambda}^\beta(\mathbb{R}^n)$ be the homogeneous Lipschitz space, which is the dual of $H^p(\mathbb{R}^n)$. While the pointwise product fg, with $f \in H^p(\mathbb{R}^n)$ and $g \in \dot{\Lambda}^\beta(\mathbb{R}^n)$ does not make sense in general, it is possible to define it in the distribution sense and to prove that it belongs to the space $L^1(\mathbb{R}^n) + H^p_{w_p}(\mathbb{R}^n)$, with w_p the weight

$$w_p(x) := \begin{cases} \dfrac{1}{[(1+|x|)^n]^{1-p}} & \text{when } n(1/p-1) \notin \mathbb{N}, \\ \dfrac{1}{[(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{when } n(1/p-1) \in \mathbb{N}. \end{cases}$$

Moreover one can find continuous bilinear operators, which may be expressed in terms of paraproducts.

This is in relation with multipliers of the (homogeneous) Lipschitz space $\dot{\Lambda}^\alpha(\mathbb{R}^n)$.

As an application, one has estimates for the div-curl product.

This is a joint work with Jun Cao, Luong Dang Ky, Liguang Liu, Dachun Yang and Wen Yuan.