Spikes in perturbative cosmology
or
Saddles and Spikes

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Relativistic cosmological modelling

A *relativistic* cosmological model involves several ingredients:

- *Kinematics*: *a 4-dimensional spacetime* (M, g_{ab}), where M is a 4-dimensional manifold and g_{ab} is a metric field with Lorentzian signature $-+++$.
- *Matter fields*: $\Rightarrow$ several, possibly interacting, stress-energy tensor T_{ab} components; modified gravity, sometimes additional "gravity" fields. Reflecting the properties of e.g. cosmological large scale structure.
- *Dynamical laws*:
  (i) Dynamical laws in a curved spacetime geometry.
  (ii) A dynamical law for the spacetime geometry.
- *Boundary* (including initial) *conditions*, e.g., assumptions regarding e.g. the existence of spacelike Cauchy hypersurfaces, compatibility with (and simplifications based on) the CMB and assumed large scale structure.
- *An ontological interpretation*.

A common lesson: *Model hierarchies*:

- Geometric symmetry hierarchies, and asymptotic geometric symmetry hierarchies. Homothetic and isometry symmetries have a special status as affine and curvature collinations — Lie contractions.
- Source hierarchies and source contractions.

Analysis of a relativistic cosmological model involves several ingredients:

- State space analysis of true degrees of freedom.
- A complete state space adapted coordinate cover, *respecting state space topology*, including those state space boundaries for which the equations can be extended.
- Local and global dynamical systems analysis.
- Physical solution space interpretation.

Example: State space analysis of $f(R) = R + \alpha R^2$, $\alpha > 0$ spatially flat Robertson-Walker cosmology

JCAP 1608 (2016) no.08, 064, Artur Alho, Sante Carloni, CU; e-Print: arXiv:1607.05715

- Constraint equation for the variables $H, R, \dot{R}$:

$$-12H\left(\dot{R} + HR + \frac{H}{2\alpha}\right) + R^2 = 0.$$

- Bring the constraint to a quadratic canonical form.
- Respect dimensional considerations: $([t],[H],[R],[\dot{R}],[\alpha]) = (L, L^{-1}, L^{-2}, L^{-3}, L^2)$.

$$H = \sqrt{\frac{\alpha}{12}}(\mathrm{t} - \mathrm{x})$$

$$\dot{R} + HR + \frac{H}{2\alpha} = \frac{1}{\sqrt{12\alpha}}(\mathrm{t} + \mathrm{x}),$$

and (where t, x and R all have dimension L^{-2})

$$-\mathrm{t}^2 + \mathrm{x}^2 + R^2 = 0,$$

- Jacobian determinant of $(H, \dot{R}, R) \to (\mathrm{t}, \mathrm{x}, R)$ is $1/6$.

- The reduced vacuum state space is a 2-dimensional double cone with a joint apex, the non-hyperbolic Minkowski fixed point.

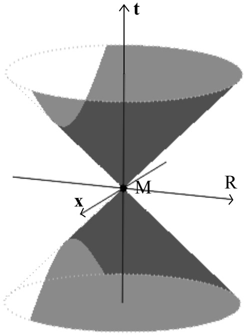


Figure: The state space light cone for $f(R) = R + \alpha R^2$, $\alpha > 0$. The shaded part denotes the state space domain of the Einstein frame, i.e., the state space of the Einstein frame is a (non-invariant) subset of that of the Jordan frame.

- $(t; H) \to -(t; H) \Rightarrow (t; \mathrm{t}, \mathrm{x}) \to -(t; \mathrm{t}, \mathrm{x})$.
- $\mathrm{t} > 0 \Rightarrow H \geq 0$.
- $H = 0 \Rightarrow \mathrm{t} = \mathrm{x}$, $R = 0$, where $\dot{R} = \mathrm{t}/\sqrt{3\alpha} > 0$.

- Consider the $\mathrm{t} > 0$ part of the state space and introduce new dimensionless bounded variables:

$$(X, S) = \left(\frac{\mathrm{x}}{\mathrm{t}}, -\frac{R}{\mathrm{t}}\right), \qquad T = \frac{1}{1 + 2\alpha \mathrm{t}} \rightarrow$$

$$X^2 + S^2 = 1.$$

  Solve the constraint:

$$X = \cos\theta, \qquad S = \sin\theta.$$

- Introduce a new dimensionless time variable $\bar{t}$:

$$\frac{dt}{d\bar{t}} = 2\sqrt{12\alpha}\, T.$$

- Dynamical system

$$T' = T(1 - T)\left[T\sin\theta + (1 - T)(1 - \cos\theta)^2\right],$$
$$\theta' = -T(3 + \cos\theta) - (1 - T)(1 - \cos\theta)\sin\theta.$$

- State space **S**, finite cylinder defined by

$$0 < T < 1.$$

- Due to regularity — extend the state space **S** to include the invariant boundaries $T = 0$ ($\mathrm{t} \rightarrow \infty \Rightarrow H \rightarrow \infty$), and $T = 1$ ($\mathrm{t} \rightarrow 0 \Rightarrow H \rightarrow 0$).

- Compactified state space $\bar{\mathbf{S}}$ of the future state space light cone + blow up the neighborhood of the non-hyperbolic Minkowski fixed point to a periodic orbit.

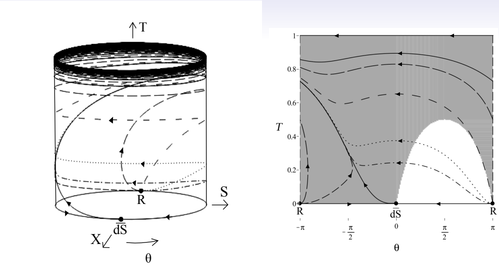


Figure: Representative solutions describing the solution space. Shaded area = the Einstein frame domain. All solutions in the Jordan frame state space end at the periodic orbit at $T = 1$, and they all originate from the fixed point R, except for 'the inflationary attractor solution' (solid line) that comes from $\bar{\mathrm{dS}}$.

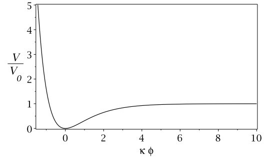


Figure: The potential $V(\phi)$ for the minimally coupled scalar field in the Einstein frame corresponding to $f(R) = R + \alpha R^2$, $\alpha > 0$.

Perturbations of ΛCDM and spike formation

Work in progress with Artur Alho and John Wainwright

- The observational success and simplicity of the flat ΛCDM models set the standard for any alternative cosmology.
- Due to explicit analytic perturbation results they serve as a comparison ground and as a test case for methods which can be used and extended to investigate more general models.

The comoving fractional density perturbation $\delta_{m,\mathrm{v}}$ satisfies

$$\delta''_{m,\mathrm{v}} + \left(2 - \tfrac{3}{2}\Omega_m\right)\delta'_{m,\mathrm{v}} - \tfrac{3}{2}\Omega_m\delta_{m,\mathrm{v}} = 0.$$

where a $'$ denotes the derivative w.r.t. the background e-fold time $N = \ln(a/a_0)$ where a is the background scale factor. Let

$$T = \frac{\lambda_m x^3}{1+\lambda_m x^3} = \Omega_\Lambda = 1 - \Omega_m,$$

where $x = a/a_0$, $\lambda_m = \Lambda/\Omega_{m0}$. Introduce the dimensionless logarithmic comoving density perturbation derivative

$$y_\mathrm{v} = \frac{\delta'_{m,\mathrm{v}}}{\delta_{m,\mathrm{v}}} \qquad \rightarrow$$

$$y'_\mathrm{v} = \tfrac{3}{2}(1-T) - \tfrac{1}{2}(1+3T)y_\mathrm{v} - y_\mathrm{v}^2,$$
$$T' = 3T(1-T).$$

Explicit solution:

$$y_{\mathrm{v}} = \frac{C_+ T^{\frac{5}{6}}(1-T)^{\frac{2}{3}}}{C_+ I + C_-} - \frac{3}{2}(1-T),$$

where $C_\pm$ are *spatial functions*,

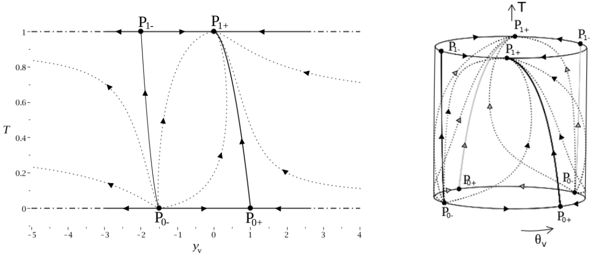
$$I(T) = I_1 - \frac{1}{2}(1-T)^{\frac{2}{3}}\, {}_2F_1(\frac{1}{6}, \frac{2}{3}; \frac{5}{3}; 1-T) = I_1 - \frac{1}{2}(1-T)^{\frac{2}{3}} \sum_{n=0}^{+\infty} \frac{(\frac{1}{6})_n(\frac{2}{3})_n}{(\frac{5}{3})_n} \frac{(1-T)^n}{n!},$$

$$I_1 = \frac{2\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}{3\sqrt{\pi}} \approx 0.5749\ldots,$$

where ${}_2F_1$ is the hypergeometric function, and $(p)_n$ the Pochhammer symbol, with $(p)_0 = 1$, and $(p)_n = p(p+1)\ldots(p+n-1)$, $n \in \mathbb{N}$.

$$y_{\rm v} = \frac{C_+ T^{\frac{5}{6}}(1-T)^{\frac{2}{3}}}{C_+ I + C_-} - \frac{3}{2}(1-T),$$

$C_- = 0$ yields the growing mode solution, $C_- = C_-^{crit} = -C_+ I_1$ gives the future saddle solution. A *past permanent spike* is generated for functions $C_\pm$ such that at a spatial point/line/surface $C_- = 0$.



(a) (b)

Figure: Solution structure for $y_{\rm v} = \delta'_{m,{\rm v}}/\delta_{m,{\rm v}} = \tan\theta$ in ΛCDM cosmology, and where T is a monotonically increasing function in the background scale factor.

Permanent spike formation (and asymptotic silence breaking)

Asymptotic isotropization in inhomogeneous cosmology W. C. Lim, H. van Elst, C. U., J. Wainwright Phys.Rev. D69 (2004) 103507 e-Print: gr-qc/0306118.

Asymptotic silence-breaking singularities, W. C. Lim, C.U., J. Wainwright, Class. Quant. Grav. 23 (2006) 2607-2630, e-Print: gr-qc/0511139

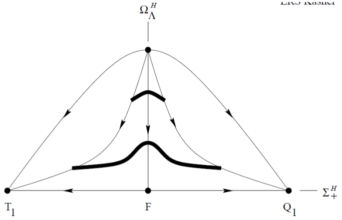
The Dynamics of inhomogeneous cosmologies, W. C. Lim, PhD Thesis, e-Print: gr-qc/0410126.

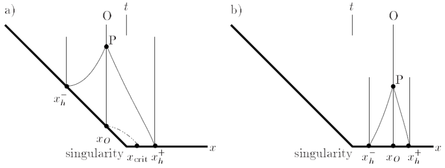
$$\mathcal{C} = \frac{C_{abcd}C^{abcd}}{H^4}, \qquad \Omega = \frac{\rho}{3H^2}$$

A special Szekeres dust solution ($\mathbf{u} = \partial/\partial t$):

$$ds^2 = -dt^2 + t^{4/3}[(a + k\,x\,t^{-1})^2 dx^2 + dy^2 + dz^2],$$

a, k, positive constants.


Ω_Λ^H
Σ_+^H
T_1
F
Q_1


a)
O
t
P
x_h^-
x_O
singularity
$x_{\rm crit}$
x_h^+
x
b)
t
O
P
singularity
x_h^-
x_O
x_h^+
x

Permanent spikes are non-generic.

Generic *oscillating spikes* arise from a 1-parameter set of transversally hyperbolic saddles, i.e., Kasner.