BKL scenario and its quantization

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What will be presented ?

Preliminaries

Classical model - Bianchi IX type

Coherent states

ACS

ACS Quantization

Quantization of BKL type model

Quantum evolution

Belinsky-Khalatnikov-Lifshitz scenario

BKL conjecture

Near a singularity, the contribution of matter to gravity becomes negligible compared with the effects of gravity as a source of further gravity, and that near a singularity, the variation of the gravitational field from one location to the next can be neglected.
- Much more important is the way gravity changes over time.

Belinsky-Khalatnikov-Lifshitz scenario

BKL conjecture

Near a singularity, the contribution of matter to gravity becomes negligible compared with the effects of gravity as a source of further gravity, and that near a singularity, the variation of the gravitational field from one location to the next can be neglected. - Much more important is the way gravity changes over time.

Generic spacetime ”behave like” spatially homogeneous spacetime close to their initial singularity

BKL dynamics

The dynamic evolution of the Universe near the initial **generic** singularity. It is described by an anisotropic, homogeneous, ”chaotic” solution to Einstein’s field equations of gravitation. The Universe is oscillating around a gravitational singularity in which time and space become equal to zero.

Singularities 1/2

Generic singularity

The singularity which exists in exact solutions of the gravitation equations.

This singularity is independent of symmetry assumptions. This means that time singularity exists not only in the special, but also in the general solutions of the Einstein equations.

Physical singularities

a) The matter is forced to be compressed to a point (a space-like singularity)
b) There are light rays which come from a region with infinite curvature (a time-like singularity)
c) ?

Singularities 2/2

V.A. Belevinski, arXiv:1404.3864v1 [gr-qc] 15 Apr 2014.

Cosmological singularity

"A cosmological singularity we mean a singularity in time, where the singular manifold is space-like, and the curvature invariants together with the invariant characteristics of any matter fields (like energy density) diverge on this manifold."

Penrose singularity

"In 1965 Roger Penrose proved ... , that under some conditions the appearance of incomplete geodesics in space-time is unavoidable. This is also a singularity but of a different type since, in general, incompleteness does not means that invariants diverge"

Generality of solutions

Criterion for generality of solutions (Landau (?))

The number of independent space coordinate functions (their number cannot be reduced by any choice of reference frame) contained in a solution.
This number is:
a) 4 for an empty (vacuum) space
b) 8 for a matter and/or radiation-filled space.

The BKL is related only to the cosmological aspect, i.e. the subject is a time singularity in the whole spacetime and not in some limited region as in a gravitational collapse of a finite body.

Bianchi spaces

A pseudo-Riemannian homogeneous space is:

a triple $(\mathrm{G}, \mathrm{M}, g)$, where G is a connected Lie group, M is a connected smooth manifold supplied with a transitive and effective action of G ,and g is a G-invariant (i. e. invariant under the action of G on M), pseudo-Remanian metric on M.

Bianchi spaces:

the triple $(\mathrm{G}, \mathrm{M}, g)$, where $\mathrm{G} = \mathrm{M} = \mathbb{R}^1 \times \mathrm{B}$ with the subgroup B of G acting transitively on 3-dimensional space-like hypersurfaces of M and the metric g is of signature (1,3). They are called Bianchi spaces since, locally, the Lie algebras corresponding to B are classified due to the Italian mathematician L. Bianchi.

Bianchi IX non-diagonal model 1/6

E. Czuchry, W. Piechocki, PRD 87, 084021 (2013).

$$ds^2 = dt^2 - \gamma_{ab}(t) e^{(a)}_{\alpha} e^{(b)}_{\beta} dx^{\alpha} dx^{\beta}$$

3 Killing vectors $e^{(a)}_{\alpha}$, $a = 1, 2, 3$; spatial metric, $\gamma_{ab}(t)$ $\alpha = 1, 2, 3$;

Redefinition of time:

$$dt = \sqrt{\gamma} d\tau$$

Equations of motion:

$$\frac{\partial^2 \ln(a)}{\partial^2 \tau^2} = \frac{b}{a} - a^2 \, , \quad \frac{\partial^2 \ln(b)}{\partial^2 \tau^2} = a^2 - \frac{b}{a} + \frac{c}{b} \, , \quad \frac{\partial^2 \ln(c)}{\partial^2 \tau^2} = a^2 - \frac{c}{b}$$

The solutions have to satisfy the constraint:

$$\frac{\partial \ln(a)}{\partial \tau} \frac{\partial \ln(b)}{\partial \tau} + \frac{\partial \ln(a)}{\partial \tau} \frac{\partial \ln(c)}{\partial \tau} + \frac{\partial \ln(b)}{\partial \tau} \frac{\partial \ln(c)}{\partial \tau} = a^2 + \frac{b}{a} + \frac{c}{b}$$

Bianchi IX non-diagonal model 2/6

Dynamical equations of the nondiagonal Bianchi IX can be obtained from the Lagrangian:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}_I}\right) - \frac{\partial L}{\partial q_I} = 0$$

where $q_I := \ln a_I,\ I = 1, 2, 3.$

$a_1 = a,\ a_2 = b,\ a_3 = c;\ a \gg b \gg c,$
are scale factors dependent only on time .
Note: $\dot{q}_I = \frac{\dot{a}_I}{a_I}$ ("Hubble parameter" type variables)

Lagrangian

$L := \dot{q}_1\dot{q}_2 + \dot{q}_1\dot{q}_3 + \dot{q}_2\dot{q}_3 + \exp(2q_1) + \exp(q_2 - q_1) + \exp(q_3 - q_2)$

Bianchi IX non-diagonal model 3/6

The conjugated momenta

$$p_I = \frac{\partial L}{\partial \dot{q}_I}$$
$$p_1 = \dot{q}_2 + \dot{q}_3 \ , \ p_2 = \dot{q}_1 + \dot{q}_3 \ , \ p_3 = \dot{q}_1 + \dot{q}_2$$

Hamiltonian

$$H_c := \frac{1}{2}\left(p_1 p_2 + p_1 p_3 + p_2 p_3\right) - \frac{1}{4}\left(p_1^2 + p_2^2 + p_3^2\right)$$
$$- \exp(2q_1) - \exp(q_2 - q_1) - \exp(q_3 - q_2)$$

which is in fact the constraint

$$H_c = 0$$

Bianchi IX non-diagonal model 4/6

Because (from eq. of motions)

$$\dot{p}_3 = \{p_3, H_c\} = \exp(q_3 - q_2) > 0$$

$t := p_3(\tau)$ can be used as the evolution parameter - $t \equiv$ "time".
The reduced dynamics with respect to new "time" t and
$H := -q_3$ gives the effective Hamiltonian:

$$H(t, q_1, q_2, p_1, p_2) = -q_2 - \ln(F(t, q_1, q_2, p_1, p_2))$$
$$F(t, q_1, q_2, p_1, p_2) := -\exp(2q_1) - \exp(q_2 - q_1)$$
$$-\frac{1}{4}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{1}{2}\left(p_1 p_2 + p_1 p_3 + p_2 p_3\right)$$

H will be quantized.

Bianchi IX non-diagonal model 5/6

There are two cases:

1. $t < 0$ (new time) – collapse of the gravitational system,
2. $t > 0$ – expansion of the gravitational system

In the following the second case is considered which implies also $p_1 > 0$ and $p_2 > 0$. The corresponding phase space:

$$\Pi = \Pi_1 \times \Pi_2 = \{(q_1, p_1) \in \mathbb{R} \times \mathbb{R}_+\} \times \{(q_2, p_2) \in \mathbb{R} \times \mathbb{R}_+\}$$

Each Π_k, $k = 1, 2$ can be identified with the manifold of the affine group Aff($\mathbb{R}$) acting on $\mathbb{R}^1$ and which is sometimes denoted as "$px + q$":

$$x' = (q, p) \cdot x = px + q, \text{ where } p > 0 \text{ and } q \in \mathbb{R}$$

This opens a possibility for quantization by coherent states.

Bianchi IX non-diagonal model 6/6

The Hamilton equations

$$\begin{aligned}
\frac{dq_1}{dt} &= \frac{\partial H}{\partial p_1} = \frac{p_1 - p_2 - t}{2F} \\
\frac{dq_2}{dt} &= \frac{\partial H}{\partial p_2} = \frac{-p_1 + p_2 - t}{2F} \\
\frac{dp_1}{dt} &= -\frac{\partial H}{\partial q_1} = \frac{-2e^{2q_1} + e^{q_2 - q_1}}{F} \\
\frac{dp_2}{dt} &= -\frac{\partial H}{\partial q_2} = 1 - \frac{e^{q_2 - q_1}}{F}
\end{aligned}$$

Critical points

$$q_1 \to -\infty \quad , \quad q_2 - q_1 \to -\infty \quad , \quad F \to 0^+$$

Coherent states 1/4

Let

1. G = a Lie group,
2. $\hat{g}$ = an unitary and irreducible representation acting in the Hilbert space $\mathcal{K}$.
3. $|\Phi_0\rangle \in \mathcal{K}$ = fixed vector, called either fiducial or cyclic vector.
4. The orbit of the fiducial vector

$$\mathcal{O}_{\Phi_0} = \{|g\rangle : \ |g\rangle = \hat{g}|\Phi_0\rangle, g \in \mathrm{G}\}$$

Coherent states 2/4

Quantum mechanical states which differ by a phase factor represent the same state.

$$|g_1\rangle \,\&\, |g_2\rangle \text{ represent the same quantum state } \Leftrightarrow$$
$$\hat{g}_2^{-1}\hat{g}_1|\Phi_0\rangle = e^{i\alpha}|\Phi_0\rangle, \quad \alpha \in \mathbb{R}$$

Generalized stationary group of the fiducial vector $|\Phi_0\rangle$

$$\mathrm{G} \supset \mathrm{G}_{\Phi_0} = \{h \in \mathrm{G} \colon \; \hat{h}|\Phi_0\rangle = e^{i\alpha(h)}|\Phi_0\rangle\}$$

$$|g_1\rangle \,\&\, |g_2\rangle \text{ represent the same quantum state } \Leftrightarrow$$
$$\hat{g}_2^{-1}\hat{g}_1 \in \mathrm{G}_{\Phi_0}$$

Coherent states 3/4

Assumptions:

1. G = a Lie group,
2. $\hat{g}$ = an unitary and irreducible representation acting in the Hilbert space $\mathcal{K}$.
3. Fiducial vector $|\Phi_0\rangle \in \mathcal{K}$

Definition of coherent states of type (G, Φ_0):

Case A: $\mathrm{G}_{\Phi_0} = \{1\}$

Def. The orbit $\mathcal{O}_{\Phi_0}$ is called the set of coherent states

Case B: $\mathrm{G}_{\Phi_0} \neq \{1\}$

Def. The set of all representatives $x \in \mathrm{G}/\mathrm{G}_{\Phi_0}$ generates the set of the vectors $|x\rangle = \hat{x}|\Phi_0\rangle$ called the coherent states.

Coherent states (Case A) 4/4

Decomposition of unity

$$\frac{1}{A_{\Phi_0}}\hat{B} = \mathbb{I}$$

where

$$\hat{B} = \int_{\mathrm{G}} d\mu_L(g)\hat{g}|\Phi_0\rangle\langle\Phi_0|\hat{g}^\dagger$$

The operator $\hat{B}$ is invariant with respect to G:

$$\hat{g}\hat{B}\hat{g}^{-1} = \hat{B}$$

Schur lemma $\Rightarrow \hat{B} \sim \mathbb{I}$.
The normalization constant A_{Φ_0}:

$$A_{\Phi_0} = \int_{\mathrm{G}} d\mu_L(g)|\langle\Phi_0|\hat{g}|\Phi_0\rangle|^2$$

Affine Coherent states 1/2

The affine phase space (Aff($\mathbb{R}$)):

$$\Pi = \Pi_1 \times \Pi_2 = \{(q_1, p_1) \in \mathbb{R} \times \mathbb{R}_+\} \times \{(q_2, p_2) \in \mathbb{R} \times \mathbb{R}_+\}$$

Multiplication law in (Aff($\mathbb{R}$)):

$$(q', p') \cdot (q, p) = (p'q + q', p'p)$$

with the unity $(0, 1)$.

Irreducible action

Action on the half-line, parametrization by points of Π_k:

$$U(q, p)\psi(x) = e^{iqx}\psi(px)\,,$$

where $|\psi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, $d\nu(x) = \frac{dx}{x}$

Affine Coherent states 2/2

Decomposition of unity, arbitrary fiducial vector $|\Phi_0\rangle$

$$\mathbb{I}[\Phi_0] = \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu_L(q,p)\, U(q,p)|\Phi_0\rangle\langle\Phi_0|U(q,p)^\dagger$$

The left invariant measure:

$$d\mu_L(q,p) = dq\frac{dp}{p^2}\,, \quad \int_{\mathrm{Aff}(\mathbb{R})} d\mu_L(q,p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq \int_0^{+\infty} \frac{dp}{p^2}$$

Fiducial vectors and $\mathbb{I}$

$|\Phi_0\rangle$ and $|\tilde{\Phi}_0\rangle$ = 2 fiducial vectors

$$\mathbb{I} = \mathbb{I}[\Phi_0] = \mathbb{I}[\tilde{\Phi}_0]$$

This allows to change representation from one to another fiducial vector.

Affine Coherent states quantization 1/2

Quantization

Def. Quantization is an appropriate mapping from classical to quantum observables.

Affine Coherent States (ACS) quantization:

$$\mathcal{F} \ni f \longrightarrow \hat{f} := \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu_L(q,p) |q,p\rangle f(q,p) \langle q,p| \in \mathcal{A}$$

where $|q,p\rangle \equiv U(q,p)|\Phi_0\rangle$, $\mathcal{F}$ is a space of real continuous functions on a phase space, and $\mathcal{A}$ is an algebra of *symmetric* operators acting in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$.

Affine Coherent states quantization 2/2

The norm of the operator $\hat{f}$:

$$\|\hat{f}\| \leq \frac{1}{A_\Phi} \int_{\mathrm{Aff}(\mathbb{R})} d\mu_L(q,p) |f(q,p)| \| |q,p\rangle\langle q,p| \| \leq$$
$$\frac{1}{A_\Phi} \int_{\mathrm{Aff}(\mathbb{R})} d\mu_L(q,p) |f(q,p)|$$

If $f \in L^1(\mathrm{Aff}(\mathbb{R}), d\mu_L(q,p))$, the operator $\hat{f}$ is bounded so it is a *self-adjoint* operator. Otherwise self-adjointness becomes an open problem.

This quantization can be applied to any type of observables including non-polynomial ones, which is of primary importance to the functional form of the Hamiltonian.

Quantization of BKL model 1/6

The Hamiltonian of classical dynamics:

$$H(t, q_1, q_2, p_1, p_2) = -q_2 - \ln(F^{(\theta)}(t, q_1, q_2, p_1, p_2))$$
$$F(t, q_1, q_2, p_1, p_2) := -\exp(2q_1) - \exp(q_2 - q_1)$$
$$-\frac{1}{4}(t - p_1 - p_2)^2 + p_1 p_2$$
$$F^{(\theta)}(t, q_1, q_2, p_1, p_2)) = \theta(F(t, q_1, q_2, p_1, p_2))F(t, q_1, q_2, p_1, p_2)$$

"Near" the gravitational singularity $q_1 \to -\infty$ and $q_2 - q_1 \to -\infty$, the terms $\exp(2q_1)$ and $\exp(q_2 - q_1)$ can be neglected:

$$F(t, q_1, q_2, p_1, p_2) \longrightarrow F_0(t, p_1, p_2) := -\frac{1}{4}(t - p_1 - p_2)^2 + p_1 p_2\,.$$

$$H \to H_0 := -q_2 - \ln F_0^{(\theta)}.$$

Quantization of BKL model 2/6

ACS quantization of the classical Hamiltonian H_0

$$\hat{H}_0(t) = \frac{1}{A^2_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})_1} d\mu_L(q_1,p_1) \int_{\mathrm{Aff}(\mathbb{R})_2} d\mu_L(q_1,p_2)$$
$$|q_1,p_1\rangle|q_2,p_2\rangle H_0(t,q_1,p_1,q_2,p_2)\langle q_1,p_1|\langle q_2,p_2|$$

"Space" representation of $\hat{H}_0$ in $L^2(\mathbb{R}^2_+, d\nu x_1, x_2)$, $d\nu(x_1,x_2) = \frac{dx_1}{x_1}\frac{dx_2}{x_2}$, can be obtained by the action

$$\hat{H}_0\Psi(t,x_1,x_2) = \int_{\mathbb{R}^2_+} d\nu(x_1,x_2)\langle x_1,x_2|\hat{H}_0(t)|x_1,x_2\rangle\Psi(t,x_1,x_2)$$

Quantization of BKL model 3/6

"Space" representation of $\hat{q}_k$ and $\hat{p}_k$ in $L^2(\mathbb{R}_+^2, d\nu(x_1,x_2))$

The generalized position operator $\hat{q}_k$, $k = 1, 2$

$$\hat{q}_k \Psi(t, x_1, x_2) = \left(-i\frac{\partial}{\partial x_k} + \frac{i}{2x_k}\right)\Psi(t, x_1, x_2) \ , \quad q \sim \ln(a) \ .$$

The generalized momentum operator $\hat{p}_k$, $k = 1, 2$

$$\hat{p}_k \Psi(t, x_1, x_2) = \left(\frac{1}{A_{\Phi_0}}\int_{\mathbb{R}_+} d\nu(p_k)|\Phi_0(p_k)|^2\right)\frac{1}{x_k}\Psi(t, x_1, x_2)$$

$$= \left(\frac{A^{(1)}_{\Phi_0}}{A_{\Phi_0}}\right)\frac{1}{x_k}\Psi(t, x_1, x_2) = C_p\frac{1}{x_k}\Psi(t, x_1, x_2) \ , \quad p \sim \frac{1}{x} \sim \frac{\dot{a}}{a} \ .$$

Quantization of BKL model 4/6

Denote by $\hat{F}_0^{(\theta)}(t)$ the quantized form of $F_0^{(\theta)}(t, p_1, p_2)$.

The spatial representation of the operator $\hat{F}_0^{(\theta)}(t)$ is the multiplication operator:

$$\hat{F}_0^{(\theta)}(t)\Psi(t, x_1, x_2) = \breve{F}_0^{(\theta)}(t, x_1, x_2)\Psi(t, x_1, x_2)$$

$$\breve{F}_0^{(\theta)}(t, x_1, x_2) = \frac{1}{A^2_{\Phi_0}} \int_{\mathbb{R}^2_+} \frac{dp_1}{p_1^2}\frac{dp_2}{p_2^2} F_0^{(\theta)}(t, \frac{p_1}{x_1}, \frac{p_1}{x_1})|\Phi_0(p_1)|^2|\Phi_0(p_2)|^2$$

$$F_0^{(\theta)}(t, p_1, p_2) := \theta(-\frac{1}{4}(t - p_1 - p_2)^2 + p_1p_2)[-\frac{1}{4}(t-p_1-p_2)^2+p_1p_2]\,.$$

Quantization of BKL model 5/6

"Space" representation of $\ln(F_0^{(\theta)}(t, p_1, p_2)) \to \hat{K}(t)$ operator in $L^2(\mathbb{R}_+^2, d\nu x_1, x_2)$

"Space" representation of $\ln(F_0^{(\theta)}(t, p_1, p_2)) \to \hat{K}(t)$

$$\hat{K}(t)\Psi(t, x_1, x_2) = K(t, x_1, x_2)\Psi(t, x_1, x_2)$$

where

$$K(t, x_1, x_2) = \frac{1}{A_{\Phi_0}^2} \int_{\mathbb{R}_+^2} \frac{dp_1}{p_1^2} \frac{dp_2}{p_2^2} \ln(F_0^{(\theta)}(t, \frac{p_1}{x_1}, \frac{p_2}{x_2}))|\Phi_0(p_1)|^2 |\Phi_0(p_2)|^2$$

$$F_0^{(\theta)}(t, p_1, p_2) := \theta(-\frac{1}{4}(t - p_1 - p_2)^2 + p_1 p_2)[-\frac{1}{4}(t - p_1 - p_2)^2 + p_1 p_2] \,.$$

Quantization of BKL model 6/6

The quantum Hamiltonian $\hat{H}_0$ corresponding to the classical H_0:

$$H_0 = -q_2 - \ln(F_0^{(\theta)}(t, p_1, p_2)) \rightarrow$$
$$\hat{H}_0 = +i\frac{\partial}{\partial x_2} - \frac{i}{2x_2} - K(t, x_1, x_2)$$

The space of quantum states $\mathcal{K}^{(\theta)}$:

$$\mathcal{K}^{(\theta)} \subset L^2(\mathbb{R}_+^2, d\nu(x_1, x_2))$$

Quantum evolution - BKL model 1/19

Classic evolution generator

Because classical variables $q_3 = H$ and $p_3 = t$ are canonically conjugated variables the operation $\{H, \cdot\}$ can be considered as the shift operator with respect to the evolution parameter $p_3 \equiv t$. This suggest that the quantum Hamiltonian can be treated as the quantum evolution generator.

Schrödinger type evolution

$$i\frac{\partial \Psi(t, x_1, x_2)}{\partial t} = \hat{H}_0 \Psi(t, x_1, x_2)$$

$$i\frac{\partial \Psi(t, x_1, x_2)}{\partial t} = \left(+i\frac{\partial}{\partial x_2} - \frac{i}{2x_2} - K(t, x_1, x_2) \right) \Psi(t, x_1, x_2)$$

Quantum evolution - BKL model 2/19

Solution of the Schrödinger equation. After "linear" substitutions

$$\alpha = t \text{ and } \beta = x_2 + t$$

one gets

$$\frac{\partial \Psi}{\partial \alpha} + \left(\frac{1}{2(-\alpha + \beta)} + K(\alpha, x_1, x_2(\alpha, \beta)) \right) \Psi = 0$$

Solution

$$\Psi(t, x_1, x_2) = \eta(x_1, x_2 + (t - t_0)) \sqrt{\frac{x_2}{x_2 + (t - t_0)}} \times$$

$$\times \exp\left(i \int_{t_0}^{t} K(t', x_1, x_2 + t - t') dt' \right), \quad t \geq t_0$$

where $\eta(x_1, x_2)$ is the initial, $t = t_0$ state. Important: $t_0 = 0$.

Quantum evolution - BKL model 3/19

Normalization condition (independence of time). The norm squared for nearly all functions is equal

$$\begin{aligned}\langle\Psi(t)|\Psi(t)\rangle &= \int_{\mathbb{R}^2_+} d\nu(x_1,x_2)\frac{x_2|\eta(x_1,x_2+t)|^2}{x_2+t} \\ &= \int_0^\infty d\nu(x_1)\int_t^\infty dx_2\frac{|\eta(x_1,x_2)|^2}{x_2}\end{aligned}$$

and it diminishes with time – unitarity crisis (?)

In addition it turns out that $\langle\Psi(t)|\hat{p}_2|\Psi(t)\rangle = \infty$

One needs to find some additional conditions for the initial state $\eta(x_1,x_2)$ to recover unitarity.

Quantum evolution - BKL model 4/19

The only family of functions which give unitary evolution is:

Solutions

$$\Psi(t, x_1, x_2) = \eta(x_1, x_2 + (t - t_0))\sqrt{\frac{x_2}{x_2 + (t - t_0)}} \times$$
$$\times \exp\left(i \int_{t_0}^{t} K(t', x_1, x_2 + t - t')dt'\right), \quad t \geq t_0,$$

where the initial function should fulfil the additional condition

$$\eta(x_1, x_2) = 0 \text{ for } x_2 < t_H.$$

where $t_H > 0$ is a parameter.

Quantum evolution - BKL model 5/19

Where is the singularity, if exists?

1. $q_1 \to -\infty \quad \Rightarrow \quad \langle \Psi | \hat{q}_1 | \Psi \rangle \to -\infty,$
2. $q_2 - q_1 \to -\infty \quad \Rightarrow \quad \langle \Psi | \hat{q}_2 - \hat{q}_1 | \Psi \rangle \to -\infty,$
3. $F_0 \to 0^+ \quad \Rightarrow \quad \langle \Psi | \hat{F}_0 | \Psi \rangle \to 0^+$

Because of F_0 the "singularity" exists only for $t = 0$

$$F_0^{(\theta)}(t=0, p_1, p_2) = \theta(-\frac{1}{4}(p_1 - p_2)^2) \left(-\frac{1}{4}\right) (p_1 - p_2)^2 = 0$$

$$\langle \Psi(t=0) | \hat{F}_0^{(\theta)}(t=0) | \Psi(t=0) \rangle = 0$$

"Generic singularity? "

The only singularity in this model can be at $t = 0$

Quantum evolution - BKL model 6/19

How to construct states near singularity?

Position eigenstates

$$\hat{q}_k \eta_{\lambda_1 \lambda_2}(x_1, x_2) = \lambda_k \eta_{\lambda_1 \lambda_2}(x_1, x_2)$$
$$\eta_{\lambda_1 \lambda_2}(x_1, x_2) = N_\eta \sqrt{x_1 x_2} e^{i(\lambda_1 x_1 + \lambda_2 x_2)}$$

Problem with orthogonality?

$$\langle \eta_{\lambda_1' \lambda_2'} | \eta_{\lambda_1 \lambda_2} \rangle = N_\eta' N_\eta \int_0^\infty dx_1 \int_0^\infty dx_2 \, e^{i(\lambda_1 - \lambda_1')x_1} e^{i(\lambda_2 - \lambda_2')x_2}$$
$$= \left(\frac{i}{(\lambda_1 - \lambda_1') + i0^+} \right) \left(\frac{i}{(\lambda_2 - \lambda_2') + i0^+} \right)$$

Quantum evolution - BKL model 7/19

How to construct states near singularity?
For Λ_1, Λ_2 approaching $(-\infty)$,

$$\phi(x_1, x_2) = \int_{-\infty}^{\Lambda_1} dx_1 \int_{-\infty}^{\Lambda_2} dx_2\, u(\Lambda_1, \Lambda_2; \lambda_1, \lambda_2)\eta_{\lambda_1\lambda_2}(x_1, x_2)$$

with $\langle\phi|\phi\rangle = 1$ for a given Λ_1, Λ_2, one can be as close as required to the position $(q_1, q_2) = (-\infty, -\infty)$.

However, not all states belonging to $L^2(\mathbb{R}_+^2, d\nu(x_1, x_2))$ can be used because they have to be in the domains of the observables $\hat{q}_k$ and $\hat{p}_k$.

Quantum evolution - BKL model 8/19

What are the conditions for states to have the observables $\hat{q}_k$, $\hat{p}_k$ and $\hat{F}_0(t)$ well determined?

1. To have $\hat{q}_k$ and $\hat{p}_k$ symmetric operators

$$\lim_{x_1 \to 0^+} \frac{1}{\sqrt{x_1}} \phi(x_1, x_2) = \lim_{x_2 \to 0^+} \frac{1}{\sqrt{x_2}} \phi(x_1, x_2) = 0$$

2. ϕ in the intersection of Hilbert spaces

$$\phi(x_1, x_2) \in L^2 \left(\mathbb{R}_+^2, dx_1/(x_1)^n \, dx_2/(x_2)^m \right),$$
$$\text{where } (n, m) = (1,1), (1,2), (2,1), (1,3), (3,1)$$

3. The "hole" condition, ($t_H > 0$ is a given constant):

$$\phi(x_1, x_2) = 0 \text{ for } x_2 \le t_H, \quad t_H > 0.$$

Quantum evolution - BKL model 9/19

Example of a class of states "close" to the singularity, $t = t_s \approx 0$.

The state with a "good" position expectation value

$$\Psi(t_s, x_1, x_2) = \begin{cases} 0, & \text{for } x_2 \leq t_H \\ v(x_1, x_2) e^{i(\Lambda_1 x_1 + \Lambda_2 x_2)}, & \text{for } x_2 > t_H. \end{cases}$$

$$v(x_1, x_2 = 0) = 0, \qquad \lim_{x_2 \to 0^+} \frac{\partial \Psi(t_s, x_1, x_2)}{\partial x_2} = 0$$

For example, the function

$$v(x_1, x_2) = \begin{cases} 0, & \text{for } x_2 \leq t_H \\ \frac{1}{\sqrt{N_v}} \left[\frac{x_1^2}{(\beta_1 + x_1)^3} \right] \left[\frac{(x_2 - t_H)^2}{(\beta_2 - t_H + x_2)^3} \right], & \text{for } x_2 > t_H. \end{cases}$$

fulfils required conditions.

Quantum evolution - BKL model 10/19

Example of a class of states "close" to the singularity, $t = t_s \approx 0$.

Position expectation value and variance

$$\langle \Psi(t_s) | \hat{q}_k | \Psi(t_s) \rangle = \Lambda_k$$

$$\mathrm{var}(\hat{q}_k)_{\Psi(t_s)} = \int_{\mathbb{R}_+^2} d\nu(x_1, x_2) \left[\frac{\partial v(x_1, x_2)}{\partial x_k} - \frac{v(x_1, v_2)}{2 x_k} \right]^2$$

It is independent of Λ_k.

$\hat{F}_0^{(\theta)}(t_s)$ expectation value

$$\langle \Psi(t_s) | \hat{F}_0^{(\theta)}(t_s) | \Psi(t_s) \rangle > 0,$$

and probably it is close to zero for $t_s \approx 0$ (in fact not required).

Quantum evolution - BKL model 11/19

Backward evolution to reach the initial state $\eta(x_1, x_2)$ at $t = 0$

Assumption: we know the state for $0 \leq t = t_s < t_H$

$$\Psi(t_s, x_1, x_2) = 0, \text{ for } x_2 \leq t_H - t_s \; ,$$

$$\Psi(t_s, x_1, x_2) = \eta(x_1, x_2 + t_s)\sqrt{\frac{x_2}{x_2 + t_s}} \times$$

$$\times \exp\left(i \int_0^{t_s} K(t', x_1, x_2 + t_s - t')dt' \right), \text{ for } x_2 \geq t_H - t_s \; .$$

$\Psi(t_s, x_1, x_2)$ is a "non-singular" state close to the singularity.

Quantum evolution - BKL model 12/19

Backward evolution to reach the initial state

Derivation of $\eta(x_1, x_2)$ at $t = 0$ (singularity)

$$\eta(x_1, x_2) = 0, \text{ for } x_2 \leq t_H \ ,$$

$$\eta(x_1, x_2) = \Psi(t_s, x_1, x_2 - t_s)\sqrt{\frac{x_2}{x_2 - t_s}} \times$$

$$\times \exp\left(-i\int_0^{t_s} K(t', x_1, x_2 - t')dt'\right), \text{ for } x_2 > t_H \ .$$

Density probability at $t = 0$ is well defined – no singularity:

$$\gamma = \delta(x_2 > t_H)\frac{|\Psi(t_s, x_1, x_2 - t_s)|^2}{x_1(x_2 - t_s)}$$

Quantum evolution - BKL model 13/19

For any arbitrary allowed evolving state

$$\Psi(t, x_1, x_2) = \delta(x_2 \geq t_H)\eta(x_1, x_2 + t)\sqrt{\frac{x_2}{x_2 + t}} \times$$
$$\times \exp\left(i \int_0^t K(t', x_1, x_2 + t - t')dt'\right)$$

the momentum p_1 is:

$< \hat{p}_1 >_{\Psi(t)}$

$$\langle \Psi(t)|\hat{p}_1|\Psi(t)\rangle = C_p \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty dx_2 \frac{|\eta(x_1, x_2)|^2}{x_1 x_2} = \text{const} \; .$$

The momentum p_1 is constant (independent of time).

Quantum evolution - BKL model 14/19

For any arbitrary allowed state $\Psi(t, x_1, x_2)$ the momentum p_2 is:

$< \hat{p}_2 >_{\Psi(t)}$

$$\langle\Psi(t)|\hat{p}_2|\Psi(t)\rangle = C_p \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty dx_2 \frac{|\eta(x_1,x_2)|^2}{x_2(x_2-t)} \ .$$

$\frac{d}{dt} < \hat{p}_2 >_{\Psi(t)}$

$$\frac{d}{dt}\langle\Psi(t)|\hat{p}_2|\Psi(t)\rangle = C_p \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty dx_2 \frac{|\eta(x_1,x_2)|^2}{x_2(x_2-t)^2} > 0 \ .$$

The momentum p_2 is growing monotonically with time. For all $t : 0 \leq t < t_H$ the momentum is finite, it does not “explode”.

Quantum evolution - BKL model 15/19

Momenta at the singularity $t = 0$

$< \hat{p}_1 >_\eta$

$$\langle \eta | \hat{p}_1 | \eta \rangle = C_p \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty dx_2 \frac{|\Psi(t_s, x_1, x_2 - t_s)|^2}{x_1 (x_2 - t_s)} < \infty \, .$$

$< \hat{p}_2 >_\eta$

$$\langle \eta | \hat{p}_2 | \eta \rangle = C_p \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty dx_2 \frac{|\Psi(t_s, x_1, x_2 - t_s)|^2}{x_2 (x_2 - t_s)} < \infty \, .$$

No “singular” behavior.

Quantum evolution - BKL model 16/19

Any arbitrary allowed state can be written as its absolute value and the phase factor:

$$\Psi(t, x_1, x_2) = u(t, x_1, x_2) e^{i\Lambda(t, x_1, x_2)} \ ,$$

then the expectation value of the position q_k (ln of scale) is:

$< \hat{q}_k >_{\Psi(t)}$

$$\langle \Psi(t) | \hat{q}_k | \Psi(t) \rangle = \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty \frac{dx_2}{x_2} \frac{\partial \Lambda(t, x_1, x_2)}{\partial x_k} u(t, x_1, x_2)^2 .$$

Quantum evolution - BKL model 17/19

Space position (ln of scale) at the singularity $t = 0$

The appropriate $u(x_1, x_2)$ and $\Lambda(x_1, x_2)$:

$u(x_1, x_2)$ and $\Lambda(x_1, x_2)$ for $t = 0$:

$$\eta(x_1, x_2) = u(x_1, x_2)e^{i\Lambda(x_1,x_2)} \ ,$$

$$\Psi(t_s, x_1, x_2 - t_s) = 0 \text{ for } x_2 \leq t_H \ ,$$

$$u(x_1, x_2)^2 = \delta(x_2 > t_H)|\Psi(t_s, x_1, x_2 - t_s)|^2 \frac{x_2}{x_2 - t_s} \ ,$$

$$\Lambda(x_1, x_2) = \arg(\Psi(t_s, x_1, x_2 - t_s)) - \int_0^{t_s} dt' K(t', x_1, x_2 - t').$$

Note: $0 \leq t_s < t_H$.

Quantum evolution - BKL model 18/19

Space position (ln of scale) at the singularity $t = 0$

$< \hat{q}_k >_\eta$

$$\langle\eta|\hat{q}_k|\eta\rangle = \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty \frac{dx_2}{x_2} \Big\{ \frac{\partial}{\partial x_k} \Big[\arg(\Psi(t_s, x_1, x_2 - t_s)) - \int_0^{t_s} dt' K(t', x_1, x_2 - t') \Big] \Big\} |\Psi(t_s, x_1, x_2 - t_s)|^2 \frac{x_2}{x_2 - t_s}.$$

$$K(t, x_1, x_2) = \frac{1}{x_1 x_2} \frac{1}{A_{\Phi_0}^2} \times \times \int_{\mathbb{R}_+^2} \frac{dp_1}{p_1^2} \frac{dp_2}{p_2^2} \ln(F_0^{(\theta)}(t, p_1, p_2)) |\Phi_0(x_1 p_1)|^2 |\Phi_0(x_2 p_2)|^2$$

Summary:

There is no singular behavior observed in the presented quantum model. The ACS quantization remove the classical singularity.

Problems

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