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# Singularity crossing in classical and quantum cosmology..

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# Introduction

- ▶ General relativity connects the geometrical properties of the spacetime to its matter content.  
Matter tells spacetime how to curve itself, the spacetime geometry tells matter how to move.
- ▶ Cosmological singularities constitute one of the main problems of modern cosmology.
- ▶ The discovery of cosmic acceleration stimulated the development of “exotic” cosmological models of dark energy; some of these models possess the so called soft or sudden singularities characterized by a finite value of the radius of the universe and its Hubble parameter.



- ▶ “Traditional” or “hard” singularities are associated with a zero volume of the universe (or of its scale factor), and with infinite values of the Hubble parameter, of the energy density and of the pressure –Big Bang and Big Crunch
- ▶ In some models interplay between the geometry and matter forces matter to change some of its basic properties, such as the equation of state for fluids and even the form of the Lagrangian.
- ▶ Tachyons (Born-Infeld fields) are natural candidates for a dark energy
- ▶ The toy tachyon model, proposed in 2004 has two particular features:  
The tachyon field transforms itself into a pseudo-tachyon field,  
The evolution of the universe can encounter a new type of singularity - the Big Brake singularity.

- ▶ The Big Brake singularity is a particular type of the so called “soft” cosmological singularities - the radius of the universe is finite, the velocity of expansion is equal to zero, the deceleration is infinite.
- ▶ The Big Brake singularity is a particular one - it is possible to cross it.
- ▶ It is possible to cross other soft singularities, sometimes matter changes its properties

- ▶ Crossing of the Big Bang - Big Crunch singularity: is it a question of a field parametrization ?
  
- ▶ What happens in anisotropic spacetimes ?
- ▶ What the quantum cosmology tells about the cosmological singularities ?

# Description of the tachyon model

The flat Friedmann universe

$$ds^2 = dt^2 - a^2(t)dl^2$$

The tachyon Lagrange density

$$L = -V(T)\sqrt{1 - \dot{T}^2}$$

The energy density

$$\rho = \frac{V(T)}{\sqrt{1 - \dot{T}^2}}$$

The pressure

$$p = -V(T)\sqrt{1 - \dot{T}^2}$$

The Friedmann equation

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \rho$$

The equation of motion for the tachyon field

$$\frac{\ddot{T}}{1 - \dot{T}^2} + 3H\dot{T} + \frac{V_{,T}}{V} = 0$$

In our model

$$V(T) = \frac{\Lambda}{\sin^2 \left[ \frac{3}{2} \sqrt{\Lambda(1+k)} T \right]} \\ \times \sqrt{1 - (1+k) \cos^2 \left[ \frac{3}{2} \sqrt{\Lambda(1+k)} T \right]},$$

where  $k$  and  $\Lambda > 0$  are the parameters of the model. The case  $k > 0$  is more interesting.

Some trajectories (cosmological evolutions) finish in an **infinite de Sitter expansion**. In other trajectories the tachyon field transforms into a **pseudotachyon** field with the Lagrange density, energy density and positive pressure:

$$L = W(T) \sqrt{\dot{T}^2 - 1},$$

$$\rho = \frac{W(T)}{\sqrt{\dot{T}^2 - 1}},$$

$$p = W(T) \sqrt{\dot{T}^2 - 1},$$

$$W(T) = \frac{\Lambda}{\sin^2 \left[ \frac{3}{2} \sqrt{\Lambda(1+k)} T \right]} \\ \times \sqrt{(1+k) \cos^2 \left[ \frac{3}{2} \sqrt{\Lambda(1+k)} T - 1 \right]}$$

What happens to the Universe after the transformation of the tachyon into the pseudotachyon ?

It encounters the **Big Brake** cosmological singularity.

# The Big Brake cosmological singularity and other soft singularities

$$t \rightarrow t_{BB} < \infty$$

$$a(t \rightarrow t_{BB}) \rightarrow a_{BB} < \infty$$

$$\dot{a}(t \rightarrow t_{BB}) \rightarrow 0$$

$$\ddot{a}(t \rightarrow t_{BB}) \rightarrow -\infty$$

$$R(t \rightarrow t_{BB}) \rightarrow +\infty$$

$$\rho(t \rightarrow t_{BB}) \rightarrow 0$$

$$p(t \rightarrow t_{BB}) \rightarrow +\infty$$

If  $\dot{a}(t_{BB}) \neq 0$  it is a more general soft singularity.



# Crossing the Big Brake singularity and the future of the universe

At the Big Brake singularity the equations for geodesics are regular, because the Christoffel symbols are regular (moreover, they are equal to zero).

Is it possible to cross the Big Brake ?

Let us study the regime of approach to the Big Brake.

On analyzing the equations of motion we find that on approaching the Big Brake singularity the tachyon field behaves as

$$T = T_{BB} + \left( \frac{4}{3W(T_{BB})} \right)^{1/3} (t_{BB} - t)^{1/3}.$$

Its time derivative  $s \equiv \dot{T}$  behaves as

$$s = - \left( \frac{4}{81W(T_{BB})} \right)^{1/3} (t_{BB} - t)^{-2/3},$$

the cosmological radius is

$$a = a_{BB} - \frac{3}{4} a_{BB} \left( \frac{9W^2(T_{BB})}{2} \right)^{1/3} (t_{BB} - t)^{4/3},$$

its time derivative is

$$\dot{a} = a_{BB} \left( \frac{9W^2(T_{BB})}{2} \right)^{1/3} (t_{BB} - t)^{1/3}$$

and the Hubble variable is

$$H = \left( \frac{9W^2(T_{BB})}{2} \right)^{1/3} (t_{BB} - t)^{1/3}.$$

All these expressions can be **continued** in the region where  $t > t_{BB}$ , which amounts to **crossing the Big Brake singularity**. Only the expression for  $s$  is singular at  $t = t_{BB}$  but this singularity is **integrable** and **not dangerous**.

Once reaching the Big Brake, it is impossible for the system to stay there because of the infinite deceleration, which eventually leads to a decrease of the scale factor. This is because after the Big Brake crossing the time derivative of the cosmological radius and Hubble variable change their signs. The **expansion** is then followed by a **contraction**, culminating in the **Big Crunch** singularity.

# Crossing of the soft singularity in the model with the anti-Chaplygin gas and dust

One of the simplest cosmological models revealing a Big Brake singularity is the model based on the anti-Chaplygin gas with an equation of state

$$p = \frac{A}{\rho}, \quad A > 0$$

Such an equation of state arises in the theory of wiggly strings ( B. Carter, 1989, A. Vilenkin, 1990).

$$\rho(a) = \sqrt{\frac{B}{a^6} - A}$$

At  $a = a_* = \left(\frac{B}{A}\right)^{1/6}$  the universe encounters the Big Brake singularity.

## The anti-Chaplygin gas plus dust

The energy density and the pressure are

$$\rho(a) = \sqrt{\frac{B}{a^6} - A} + \frac{M}{a^3}, \quad p(a) = \frac{A}{\sqrt{\frac{B}{a^6} - A}}.$$

Due to the dust component, the Hubble parameter has a non-zero value at the encounter with the singularity, therefore the dust implies further expansion. With continued expansion however, the energy density and the pressure of the anti-Chaplygin gas would become ill-defined.

## Change of the equation of state at soft singularity crossings

The **abrupt** transition from the expansion to the contraction of the universe does not look natural. There is an **alternative/complementary** way of resolving the paradox.

One can try to change the equation of state of the anti-Chaplygin gas on passing the soft singularity.

There is some analogy between the transition from an expansion to a contraction of a universe and the perfectly elastic bounce of a ball from a wall in classical mechanics.

There is also an abrupt change of the direction of the velocity (momentum).

However, we know that in reality the velocity is changed **continuously** due to the deformation of the ball and of the wall.

The pressure of the anti-Chaplygin gas

$$p = \frac{A}{\sqrt{\frac{B}{a^6} - A}}$$

tends to  $+\infty$  when the universe approaches the soft singularity.

Requiring the expansion to continue into the region  $a > a_S$ , while changing minimally the equation of state, we assume

$$p = \frac{A}{\sqrt{\left| \frac{B}{a^6} - A \right|}},$$

$$p = \frac{A}{\sqrt{A - \frac{B}{a^6}}}, \text{ for } a > a_S.$$

This implies the energy density

$$\rho = -\sqrt{A - \frac{B}{a^6}}.$$



The anti-Chaplygin gas transforms itself into Chaplygin gas with negative energy density.  
The pressure remains positive, expansion continues.  
The spacetime geometry remains continuous.  
The expansion stops at  $a = a_0$ , where

$$\frac{M}{a_0^3} - \sqrt{A - \frac{B}{a_0^6}} = 0.$$

Then the contraction of the universe begins. At the moment when the energy density of the Chaplygin gas becomes equal to zero (again a soft singularity), the Chaplygin gas transforms itself into the anti-Chaplygin gas and the contraction continues culminating in an encounter with the Big Crunch singularity  $a = 0$ .

# Crossing the Big Brake singularity and the future of the universe in the tachyon model in the presence of dust.

What happens with the Born-Infeld type pseudo-tachyon field in the presence of a **dust** component? Does the universe still run into a soft singularity?

Yes !

$$T = T_S \pm \sqrt{\frac{2}{3H_S}} \sqrt{t_S - t}, \quad H_S = \sqrt{\frac{\rho_{m,0}}{a_S^3}}.$$

A pseudo-tachyon field with a constant potential is equivalent to the anti-Chaplygin gas.

To the change of the equation of state of the anti-Chaplygin gas corresponds the following transformation of the Lagrangian of the pseudo-tachyon field:

$$L = W_0 \sqrt{\dot{T}^2 + 1},$$

$$p = W_0 \sqrt{\dot{T}^2 + 1}$$

$$\rho = -\frac{W_0}{\sqrt{\dot{T}^2 + 1}}.$$

It is a new type of Born-Infeld field, which we may call “quasi-tachyon”.

For an **arbitrary** potential the Lagrangian reads

$$L = W(T)\sqrt{\dot{T}^2 + 1}, \quad a > a_S$$

$$\frac{\ddot{T}}{\dot{T}^2 + 1} + 3H\dot{T} - \frac{W_{,T}}{W} = 0,$$

$$\rho = -\frac{W(T)}{\sqrt{\dot{T}^2 + 1}},$$

$$p = W(T)\sqrt{\dot{T}^2 + 1}.$$

In the vicinity of the soft singularity the friction term  $3H\dot{T}$  in the equation of motion dominates over the potential term  $W_{,T}/W$ . Hence, the dependence of  $W(T)$  on its argument is not essential and a pseudo-tachyon field approaching this singularity behaves like one with a **constant** potential. Thus, it is reasonable to assume that upon crossing the soft singularity the pseudo-tachyon transforms itself into a quasi-tachyon for any potential  $W(T)$ .

## Big Bang – Big Crunch crossing ?

- ▶ The idea that the Big Bang - Big Crunch singularity can be crossed appears very counterintuitive.
- ▶ Some approaches to the description of this crossing were elaborated during the last decade (I. Bars, S.H. Chen, P.J. Steinhardt and N. Turok, C. Wetterich, P. Dominis Prester).
- ▶ There is an analogy with the horizon which arises due to a certain choice of the spacetime coordinates: the singularity arises because of some choice of the field parametrization.

- ▶ On choosing some convenient field parametrization one can provide a matching between the characteristics of the universe before and after the singularity crossing.
- ▶ Analogy to the Kruskal coordinates for the Schwarzschild metric.
- ▶ On choosing appropriate combinations of the field variables we can describe the passage through the Big Bang - Big Crunch singularity, but this does not mean that the presence of such a singularity is not essential. Indeed, extended objects cannot survive this passage.



# Friedmann cosmology in the presence of a scalar field: Einstein frame versus Jordan frame

$$S = \int d^4x \sqrt{-g} \left[ U(\sigma) R - \frac{1}{2} g^{\mu\nu} \sigma_{,\mu} \sigma_{,\nu} + V(\sigma) \right]$$

Conformal coupling

$$U(\sigma) = U_0 - \frac{1}{12} \sigma^2$$

A conformal transformation of the metric

$$g_{\mu\nu} = \frac{U_1}{U} \tilde{g}_{\mu\nu},$$

A new scalar field  $\phi$ :

$$\frac{d\phi}{d\sigma} = \frac{\sqrt{U_1(U + 3U'^2)}}{U} \Rightarrow \phi = \int \frac{\sqrt{U_1(U + 3U'^2)}}{U} d\sigma.$$

$$\phi = \sqrt{3U_1} \ln \left[ \frac{\sqrt{12U_0 + \sigma}}{\sqrt{12U_0 - \sigma}} \right]$$

$$\sigma = \sqrt{12U_0} \tanh \left[ \frac{\phi}{\sqrt{12U_1}} \right].$$

The action then becomes the action for a **minimally coupled** scalar field:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ U_1 R(\tilde{g}) - \frac{1}{2} \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + W(\phi) \right],$$

$$W(\phi) = \frac{U_1^2 V(\sigma(\phi))}{U^2(\sigma(\phi))}.$$

This is called the transformation from the **Jordan frame** to the **Einstein frame**.

In a flat Friedmann universe

$$ds^2 = N^2 d\tau^2 - a^2 dl^2,$$

$$d\tilde{s}^2 = \tilde{N}^2 d\tau^2 - \tilde{a}^2 dl^2.$$

$$\tilde{N} = \sqrt{\frac{U}{U_1}} N, \quad \tilde{a} = \sqrt{\frac{U}{U_1}} a, \quad t = \int \sqrt{\frac{U_1}{U}} d\tilde{t},$$

where  $t$  and  $\tilde{t}$  are the **cosmic time** parameters in the Jordan and the Einstein frames.

$$a = \tilde{a} \sqrt{\frac{U_1}{U_0}} \cosh \left( \frac{\phi}{\sqrt{12U_1}} \right).$$

In the vicinity of the singularity in the **Einstein frame**:

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}} \rightarrow 0, \text{ when } \tilde{t} \rightarrow 0.$$

However, in the **Jordan frame**:

$$a \sim \tilde{t}^{\frac{1}{3}} \left( \tilde{t}^{\frac{1}{3}} + \tilde{t}^{-\frac{1}{3}} \right) \rightarrow \text{const} \neq 0.$$

Meanwhile, the scalar field  $\sigma$  crosses the value  $\pm\sqrt{12U_0}$  and the coupling function  $U$  changes its sign.

Thus, the evolution in the **Jordan frame** is **regular**, and we can use this fact to describe the **crossing** of the Big Bang - Big Crunch singularity in the **Einstein frame**.

If one considers the **expansion** of the universe from the Big Bang with normal gravity driven by the standard scalar field, the continuation **backward** in time shows that it was preceded by the **contraction** towards a Big Crunch singularity in the **antigravity** regime, driven by a **phantom** scalar field with a negative kinetic term.

The possibility of a **change of sign of the effective gravitational constant** in the model with a conformably coupled scalar field was analyzed in **1981** by **A. Starobinsky**.

It was shown that in a **homogeneous and isotropic** universe, one can indeed cross the point where the effective gravitational constant changes sign.

However, the presence of **anisotropies** changes the situation: these anisotropies **grow indefinitely** when this constant is equal to zero.

# Singularity crossing in a Bianchi - I universe

$$d\tilde{s}^2 = \tilde{N}(\tau)^2 d\tau^2 - \tilde{a}^2(\tau)(e^{2\beta_1(\tau)} dx_1^2 + e^{2\beta_2(\tau)} dx_2^2 + e^{2\beta_3(\tau)} dx_3^2),$$

$$ds^2 = N(\tau)^2 d\tau^2 - a^2(\tau)(e^{2\beta_1(\tau)} dx_1^2 + e^{2\beta_2(\tau)} dx_2^2 + e^{2\beta_3(\tau)} dx_3^2),$$

$$\beta_1 + \beta_2 + \beta_3 = 0.$$

$$\dot{\beta}_i = \frac{\beta_{i0}}{\tilde{a}^3}, \quad \theta_0 = \beta_{10}^2 + \beta_{20}^2 + \beta_{30}^2.$$

$$\dot{\phi} = \frac{\phi_0}{\tilde{a}^3}, \quad \phi = \frac{\phi_0}{\left(\frac{3\theta_0}{2} + \frac{3\phi_0^2}{4U_1}\right)^{\frac{1}{2}}} \ln \tilde{t}.$$



In the vicinity of the singularity in the Einstein frame

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}}.$$

In the Jordan frame

$$a \sim \tilde{t}^{\frac{1}{3}}(\tilde{t}^{\gamma} + \tilde{t}^{-\gamma}) \rightarrow 0,$$

because

$$\gamma = \frac{\phi_0}{3\sqrt{\phi_0^2 + 2\theta_0 U_1}} < \frac{1}{3}.$$

Thus, one also encounters the Big Bang singularity in the Jordan frame.

# Mixing between geometrical and matter degrees of freedom and the singularity crossing

The Friedmann model with a massless scalar field can be described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2,$$

where

$$x = \frac{4\sqrt{U_1}}{\sqrt{3}} \tilde{a}^{\frac{3}{2}} \cosh \frac{\sqrt{3}}{4\sqrt{U_1}} \phi, \quad y = \frac{4\sqrt{U_1}}{\sqrt{3}} \tilde{a}^{\frac{3}{2}} \sinh \frac{\sqrt{3}}{4\sqrt{U_1}} \phi,$$

and the Friedmann equation is

$$\dot{x}^2 - \dot{y}^2 = 0.$$

Inversely,

$$\tilde{a}^3 = \frac{3(x^2 - y^2)}{16U_1},$$

$$\phi = \frac{4\sqrt{U_1}}{\sqrt{3}} \operatorname{arctanh} \frac{x}{y}.$$

Initially

$$x > |y|.$$

The solution

$$x = x_1 \tilde{t} + x_0, \quad y = y_1 \tilde{t} + y_0, \quad x_1^2 = y_1^2.$$

Choosing the constants as

$$x_0 = y_0 = A > 0, \quad x_1 = -y_1 = B > 0,$$

we have

$$\tilde{a}^3 = \frac{3AB\tilde{t}}{4U_1}.$$

We can make a **continuation** in the plane  $(x, y)$ , to  $x < |y|$  or, in other words, to  $\tilde{t} < 0$ . Such a continuation implies an **antigravity** regime and the transition to the **phantom** scalar field, just as in the more complicated schemes, discussed before.

How can we generalize these considerations to the case when the **anisotropy** term is present ?

$$L = \frac{1}{2}\dot{r}^2 - \frac{1}{2}r^2(\dot{\varphi}^2 + \dot{\varphi}_1^2 + \dot{\varphi}_2^2),$$

$$\varphi_1 = \sqrt{\frac{3}{8}}\alpha_1, \quad \varphi_2 = \sqrt{\frac{3}{8}}\alpha_2,$$

$$\beta_1 = \frac{1}{\sqrt{6}}\alpha_1 + \frac{1}{\sqrt{2}}\alpha_2, \quad \beta_2 = \frac{1}{\sqrt{6}}\alpha_1 - \frac{1}{\sqrt{2}}\alpha_2, \quad \beta_3 = -\frac{2}{\sqrt{6}}\alpha_1.$$

We can again consider the plane  $(x, y)$  as

$$x = r \cosh \Phi,$$

$$y = r \sinh \Phi,$$

where a new **hyperbolic** angle  $\Phi$  is defined by

$$\Phi = \int d\tilde{t} \sqrt{\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}^2}.$$

We have reduced a **four-dimensional** problem to the old **two-dimensional** one, on using the fact that the variables  $\alpha_1, \alpha_2$  and  $\phi$  enter into the equation of motion for the scale factor  $\tilde{a}$  only through the squares of their time derivatives.

The behaviour of the scale factor before and after the crossing of the singularity can be matched by using the transition to the new coordinates  $x$  and  $y$ , which mix **geometrical** and **scalar field** variables in a particular way.

To describe the behaviour of the **anisotropic** factors it is enough to fix the constants  $\beta_{i0}$ .

# Quantum cosmology and singularities

Speaking about quantum cosmology and singularities people mean two **different** things:

**Modification** of the **Friedmann** equation.

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \rho_{\text{matter}} + \rho_{\text{quantum corrections}}.$$

**Vanishing** of the **quantum state of the universe**.

$$\Psi(\text{geometry} + \text{matter})_{\text{geometry is singular}} = 0.$$

Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = 0.$$

Where is the **time** ?

What is the **probability** ?

A time can be defined as a certain function of **geometrical** variables.

After that the wavefunction describing **matter** variables satisfies an effective Schrödinger equation. The singularity is associated with such values of the matter variables when this singularity arises in the **classical** theory.

Our analysis of some simple models tells that the probability of the arising of **soft** singularities is not suppressed by the wave function of the universe, while the probability of **Big Bang – Big Crunch** singularity tends to zero.



## Conclusions and discussion

- ▶ General relativity contains many surprises concerning relations between matter and geometry. It is enough to take it seriously.
- ▶ There is no need to be afraid of singularities!