NATIONAL CENTRE FOR NUCLEAR RESEARCH
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Phase space formulation of quantum mechanics: coherent states and coherent spaces

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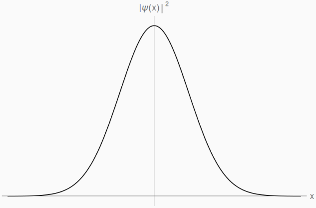
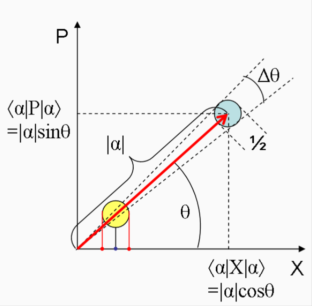
"Schrödinger" coherent state

The standard coherent state introduced in 1926 by Erwin Schrödinger.

$$|q,p\rangle = e^{(q+ip)\hat{a}^\dagger - (q-ip)\hat{a}}|0\rangle$$

- Eigenstate of the anihilation operator $\hat{a}|q,p\rangle = (q+ip)|q,p\rangle$
- Created by action of Weyl-Heisenberg group $|q,p\rangle = \hat{D}(q,p)|0\rangle$
- Saturate the Heisenberg inequality $\langle\Delta\hat{Q}\rangle_{q,p}\langle\Delta\hat{P}\rangle_{q,p} = \frac{1}{2}$
- Overcomplete $\mathbb{I} = \frac{1}{\pi}\int dqdp|q,p\rangle\langle q,p|$





Generalized coherent states

We would like to generalize "Schrödinger" coherent state using the overcompletness property and connection to the symmetry groups

- Overcomplete Family of States

$$\vec{\alpha} \mapsto |\vec{\alpha}\rangle \in \mathcal{H}$$

$$\mathbb{I} = \int d\vec{\alpha} |\vec{\alpha}\rangle\langle\vec{\alpha}|$$

- One can connect the classical labels with some physical observables
- Unitary Irreducible Representation $U(\chi)$ of canonical transformations symmetry group on phase space χ

$$\chi \ni \vec{\alpha} \mapsto |\vec{\alpha}\rangle = U(\chi)|\phi_0\rangle$$

$|\phi_0\rangle$ is called fiducial vector

Weyl-Heisenberg and Affine coherent states

Weyl-Heisenberg coherent states $\quad q \in \mathbb{R},\ p \in \mathbb{R},\ \theta \in \mathbb{R}$

W-H algebra:

$(\theta', q', p') \cdot (\theta, q, p) = (\theta' + \theta + \frac{1}{2}(p'q - pq'), q' + q, p' + p)$

Unitary irreducible representation:

$$\langle x| \underbrace{U(\theta, q, p)|\phi_0\rangle}_{|\theta,q,p\rangle} = e^{i\theta} e^{ip(x-1/2)} \phi_0(x - q) \ or\ D(q, p)|\phi_0\rangle = |q, p\rangle$$

Affine coherent states $\quad q \in \mathbb{R}_+,\ p \in \mathbb{R}$

Affine algebra:

$(q', p') \cdot (q, p) = (q'q, p/q' + p')$

Unitary irreducible representation:

$$\langle x| \underbrace{U(q, p)|\phi_0\rangle}_{|q,p\rangle} = \frac{e^{ipx}}{\sqrt{q}} \phi_0\left(\frac{x}{q}\right)$$

Semi-classical analysis with coherent states

- Restriction of the Hilbert space to coherent states subspace

$$S = \int dt \langle \psi | i\partial_t - \hat{H} | \psi \rangle \to S_R = \int dt \langle q, p | i\partial_t - \hat{H} | q, p \rangle = \int dt \{ \dot{q} p - H^s(q, p) \}$$

- We have hamilton-like equations for labels (q, p)

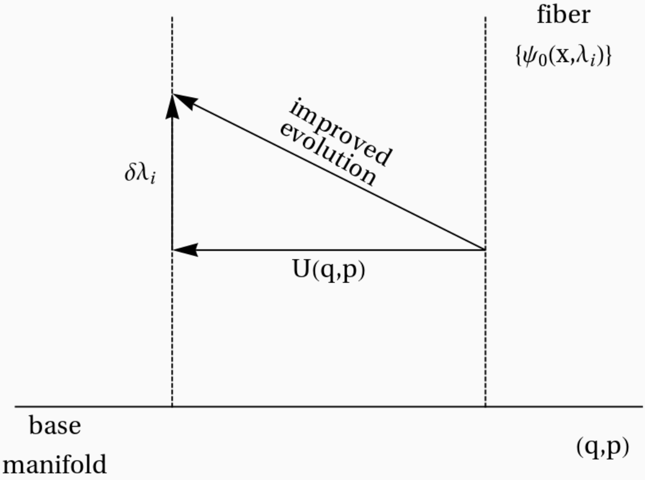
$$\dot{q} = \frac{\partial H^s(q, p)}{\partial p}, \quad \dot{p} = -\frac{\partial H^s(q, p)}{\partial q}$$

- To get physical interpretation for (q, p) choose ϕ_0 such that

$$\langle q, p | \hat{Q} | q, p \rangle = q \Rightarrow \langle \phi_0 | \hat{Q} | \phi_0 \rangle = 0 \ (1)$$

$$\langle q, p | \hat{P} | q, p \rangle = p \Rightarrow \langle \phi_0 | \hat{P} | \phi_0 \rangle = 0 \ (0)$$

Flow in the coherent space



Hamiltonian formulation

- Now we allow the evolution of the shape of the fiducial vector $\phi_0(\{\lambda\})$

$$S_E = \int dt\{\dot{q}p - \dot{\lambda}_i G^i - H^s(q, p, \{\lambda\})\}$$

- The physical restrictions on fiducial vector become constraints

$$H_{TOT} = H^s + c_2\langle\phi_0(\{\lambda\})|\ \hat{Q}\ |\phi_0(\{\lambda\})\rangle + c_3\langle\phi_0(\{\lambda\})|\ \hat{P}\ |\phi_0(\{\lambda\})\rangle$$

- The constraints are of second kind
- The improved hamilton-like equations

$$\dot{q} = \{q, H_{TOT}\}, \quad \dot{p} = \{p, H_{TOT}\}, \quad \dot{\lambda}_a = \{\lambda_a, H_{TOT}\}$$

- Increasing the number of $\lambda' s$ improve accuracy of the analysis. In the limit $\#_\lambda \mapsto \infty$ we get full quantum mechanics in ∞-dimensional phase space

Linear approach to hamiltonian formulation

Assume a basis for the fiducial vector

$$|\phi_0\rangle = \sum_i \lambda_i |e_i\rangle$$

The constraints

$$\sum_i |\lambda_i|^2 = 1,\ \lambda_i^* Q_{ij} \lambda_j = 0\ (1),\ \lambda_i^* P_{ij} \lambda_j = 0$$

The total hamiltonian

$$H_{TOT} = H^s + c_1 \sum_i |\lambda_i|^2 + c_2 \lambda_i^* Q_{ij} \lambda_j + c_3 \lambda_i^* P_{ij} \lambda_j$$

The equations of motion

$$\dot{q} = \frac{\partial H_{TOT}}{\partial p}$$

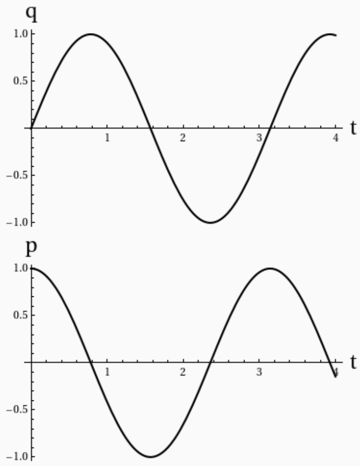
$$\dot{p} = -\frac{\partial H_{TOT}}{\partial q}$$

$$i\dot{\lambda}_i = \frac{\partial H_{TOT}}{\partial \lambda_i^*}$$

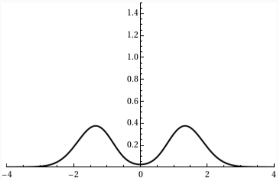
W-H example: harmonic oscillator

Particle trapped in the quadratic potential

$$\hat{H} = \hat{p}^2 + \hat{x}^2$$



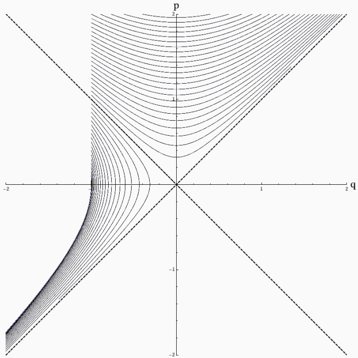
- Due to the choice of fiducial basis problem is solved exactly
- The classical degrees of freedom "decouple" from quantum ones
- State: $|\phi_0(t=0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle$



W-H example: anti-harmonic oscillator

Scattering on the inverse harmonic potential

$$\hat{H} = \hat{p}^2 - \hat{x}^2$$



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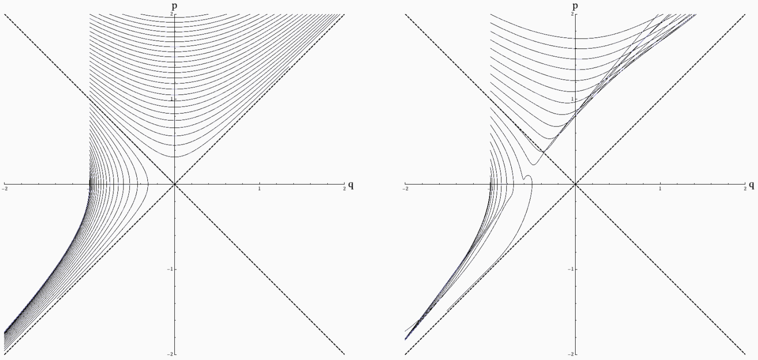
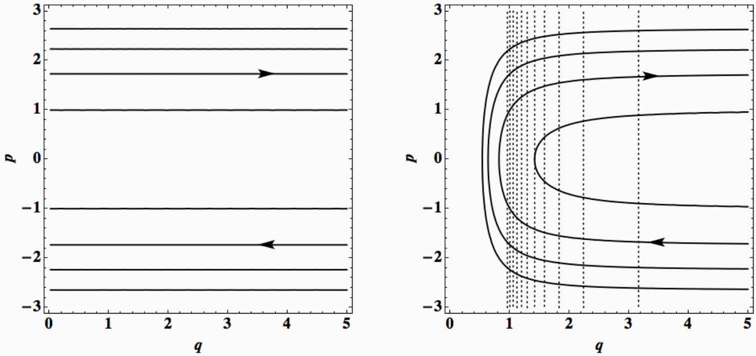


Affine example: free particle on the half-line

$$\hat{H} = \hat{p}^2,\ \ p \in \mathbb{R},\ \ q \in \mathbb{R}_+$$

$$H^s = p^2 + \frac{K}{q^2}$$



Affine example: FRW universe and the big bounce

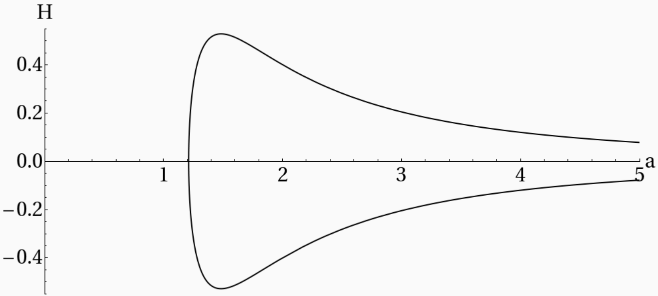
- Hamiltonian constraint:

$$\mathcal{C} = Nq^{-1}\left(-p^2 + p_T\right) \Rightarrow H = p^2,$$

where $p = a^2 H$

- Quantum hamiltonian

$$\hat{H} = -\Delta \Rightarrow H^s = p^2 + \frac{K(\lambda_i)}{q^2}$$



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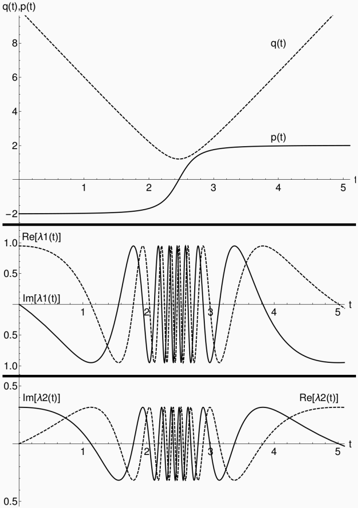

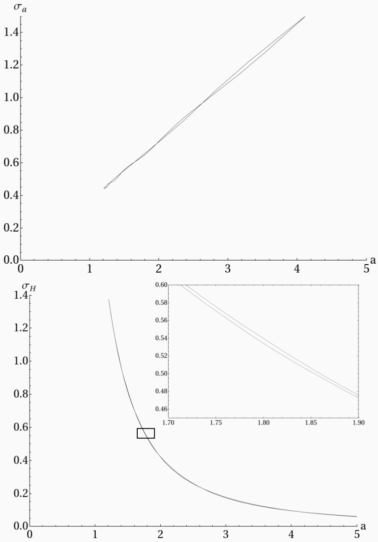

Summary

What is done:

- Construction of the extended coherent states with dynamical fiducial
- Hamiltonian formulation of quantum mechanics
- Semi-classical theory when the fiducial basis is truncated
- Simple tests with trapping and scattering potentials
- Hints of non-symmetric Big Bounce scenarios

What schould be done soon:

- Further investigation of properties of big bounce scenario
- Research on emergence of a classical universe
- Formulation of the framework for primordial gravitational waves

P. Malkiewicz, AM, H. Bergeron; arXiv:1712.04813