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On the nature of spacetime singularities

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What does it mean that a spacetime is singular?

- various attempts had been made to get rid of the uncomfortable feelings created by **the singularity theorems of Penrose and Hawking** predicting the existence of spacetime singularities
  - spacetimes describing the expanding universe and the gravitational collapse of stars are causal geodesically incomplete
- immediate suggestions were made to resolve the anticipated occurrence of curvature blow up by using
  - various theories alternative to GR, or
  - quantized versions of general relativity
- BUT, do we have a proof guaranteeing that the anticipated curvature blow up or any other possible violent behavior do really happen?
- NO, NOT AT ALL !!! NOT YET !!!
- This talk is to outline of an argument that is expected to feel up the corresponding gap, and to provide real motivations for those who are not happy with the existence of singularities showing up in GR
- the entire talk will be completely classical, no attempts are made to use alternative or quantized theories

What do we know about spacetime singularities?

- singularity $\Longleftrightarrow$ synonym of incompleteness
- **Kruskal-Szekeres:** global extension of Schwarzschild —— some parts are missing indicated by incompleteness of certain geodesics
- it is not exactly that type as in the known examples the incomplete causal geodesics do really **terminate** on some curvature singularities
- geodesic incompleteness, however, does not imply that anything dramatic could happen —— remove e.g. a point from a complete spacetime
- how could we show (if at all) that something violent is to happen **"there"** —— **where?**
- let us pretend for the moment that we (for some miracle) have a spacetime that is guaranteed to be maximal, and also it is a Cauchy development of some regular initial data
- **hey**, the existence of such maximal Cauchy development is ensured by a seminal theorem of **Choquet-Bruhat and Geroch (1969)** although it assumes smoothness of all the involved structures

The cosmic censor conjecture again?

- but there are examples with **smooth Cauchy horizons** allowing the existence of incomplete causal geodesics which can be continued in a spacetime **containing as a part** the maximal Cauchy development
- **Penrose's strong cosmic censorship conjecture** says that the maximal globally hyperbolic developments of **generic initial data** is never part of a larger spacetime.
- it begins to dawn that there is **a possible reasoning**: if nothing violent happens at the **"ideal endpoint"** of an incomplete (for definiteness assume that it is) timelike geodesic then there could exist an extension. If, however, the spacetime is maximal this is not allowed to happen which proves, by contradiction, that something violent should happen.
- this would immediately offer **the possibility to strengthen the conclusion of the singularity theorems** on the cost of assuming that the strong cosmic censorship conjecture holds, and that there is a meaningful notion of maximal Cauchy development in the considered case

What we are supposed to have by now?

- Summing up:
  - singularity theorem by **Hawking and Penrose**
  - maximal Cauchy development by **Choquet-Bruhat and Geroch**
  - strong cosmic censorship conjecture by **Penrose**
  - we also need some results on spacetime extensions **???**
    (...the original spacetime should be part of a larger one...)

What does the word extending mean?

- any spacetime is locally $\mathbb{R}^n$ (...differentiable manifold...)
  - how would we extend a function given on a compact subset in $\mathbb{R}^n$?
  - this had been studied by **Hassler Whitney ∼1930**:
    * Question: Let $\mathcal{F}$ be a real-valued function defined on a compact subset $\mathscr{A}$ of $\mathbb{R}^n$. How can we tell whether there exists $\widetilde{\mathcal{F}} \in C^m(\mathbb{R}^n)$ with $\widetilde{\mathcal{F}} = \mathcal{F}$ on $\mathscr{A}$?
    * see also **Charles Fefferman (2005)**

Hassler Whitney's extension result (1934!)

- **Definition:** a point set $\mathscr{A} \subset \mathbb{R}^n$ is said to possess **the property** $\mathscr{P}$ if there is a positive real number ω such that for any two points x and y of $\mathscr{A}$ can be joined by a curve in $\mathscr{A}$ of length $L \leq \omega \cdot \rho(x, y)$, where $\rho(x, y)$ denotes the Euclidean distance of the points $x, y \in \mathbb{R}^n$.

- **Theorem [Whitney]:** Assume that $\mathscr{A} \subset \mathbb{R}^n$ has property $\mathscr{P}$, and let $\mathcal{F}(x^1, ..., x^n)$ be of class C^m in $\mathscr{A}$, for some positive integer $m \in \mathbb{N}$. Suppose that for all $\ell \leq m$, with $\ell \in \mathbb{N}$, all the ℓ^{th} order derivatives $\partial_{x^1}^{\ell_1} \cdots \partial_{x^n}^{\ell_n} \mathcal{F}$, with $\ell_1 + \cdots + \ell_n = \ell$, can be defined on the boundary $\partial \mathscr{A}$ of $\mathscr{A}$ such that they are continuous in $\overline{\mathscr{A}} = \mathscr{A} \cup \partial \mathscr{A}$. Then there exists an extension $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ such that $\widetilde{\mathcal{F}}$ is of class C^m throughout $\mathbb{R}^n$. Moreover, the extension $\widetilde{\mathcal{F}}$ can be chosen such that it is smooth (or even it can be guaranteed to be analytic) in $\mathbb{R}^n \setminus \overline{\mathscr{A}}$.
  - property $\mathscr{P}$ may seem to be obviously possessed but it helps to think about its importance if one chooses a subregion in $\mathbb{R}^2$ between two properly arranged spirals both having their limits on the unit circle

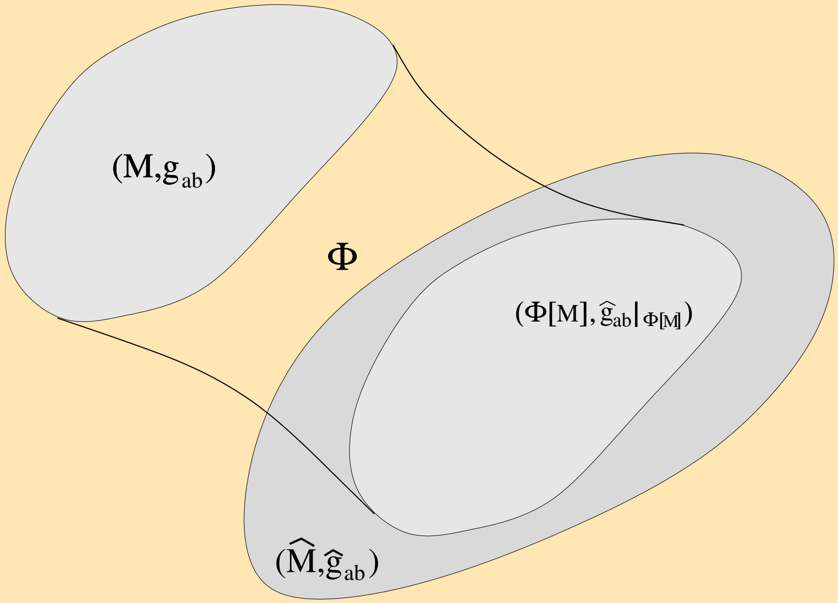
Do we really want to extend a spacetime?

Spacetime (M, g_{ab})**:** M is a smooth, paracompact, connected, orientable manifold endowed with a smooth metric g_{ab} of Lorentzian signature

Definition: Let (M, g_{ab}) and $(\widehat{M}, \widehat{g}_{ab})$ be of class C^X (smooth?) spacetimes

- a map $\Phi : (M, g_{ab}) \to (\widehat{M}, \widehat{g}_{ab})$ is said to be an **isometric imbedding** if Φ is a C^X-diffeomorphism between M and $\Phi[M] \subset \widehat{M}$ such that it carries the metric g_{ab} into $\widehat{g}_{ab}|_{\Phi[M]}$, i.e. $\Phi^* g_{ab} = \widehat{g}_{ab}|_{\Phi[M]}$

- the spacetime $(\widehat{M}, \widehat{g}_{ab})$ is called to be **a** C^X **extension** of the spacetime (M, g_{ab}) if $\Phi[M]$ is a **proper subset** of $\widehat{M}$



Do we really want to extend a spacetime?

- a spacetime (M, g_{ab}) tacitly is always assumed to represents all the events compatible with the history of the investigated physical system
- if we do extend a spacetime
  - the construction should only refer to the existing mathematical structures
  - smoothness is used frequently but only for mathematical conveniences
- nevertheless, on physical grounds it seems to be reasonable to admit metrics and other fields which are less well behaved than smooth
- the wider the class of metrics allowed the wider will be the class of material sources which can be described within the selected setup
- which is the most suitable differentiability class the fundamental field variables should belong to?

Is there any appropriate differentiability class?

- smooth or even C^k, for some $k \geq 2$, may be too much to be required
- **physical theory:** the field equations and their solutions or the possible breakdown of the field equations what is of fundamental interest for us
- **Geroch and Traschen: (1987)** the widest possible class of metrics such that the Riemann, Einstein and Weyl tensors make sense (at least) as distributions —— the space of
  - **regular metrics:**
    - g_{ab} locally bounded
    - with locally bounded inverse g^{ab}
    - the weak first derivatives "$\partial_c g_{ab}$" are locally square-integrable
  - C^0 **regular metrics:**
    - if a regular metric is **continuous** it can be approximated by sequences of smooth metrics $\{g^{(i)}{}_{ab}\}$ such that
    - the sequence of the curvature tensors $\{R^{(i)}{}_{abc}{}^d\}$, determined by that of the metrics $\{g^{(i)}{}_{ab}\}$, do converge in L_2-norm to the curvature distribution $R_{abc}{}^d$ of the continuous regular metric g_{ab}

The C^0 regular metrics may be still too rough

- though the wider the class of metrics allowed the wider is the class of mater sources that can be described within the selected framework

What about the compromise using C^{1-} metrics?

- $C^{1-} \subset C^0$ *G-T regular* $\subset$ *G-T regular*
- Examples of solutions with regular C^{1-} metrics include:
  - gravitational shock waves
  - thin mass shells
  - solutions containing pressure free matter where the geodesic flow lines have two- or three-dimensional caustics
  - the presence of incomplete causal geodesics within this class of spacetimes definitely indicates very serious break down of physics
- on top of this a number of mathematical conveniences are immediately guaranteed for locally Lipschitz, C^{1-}, metrics

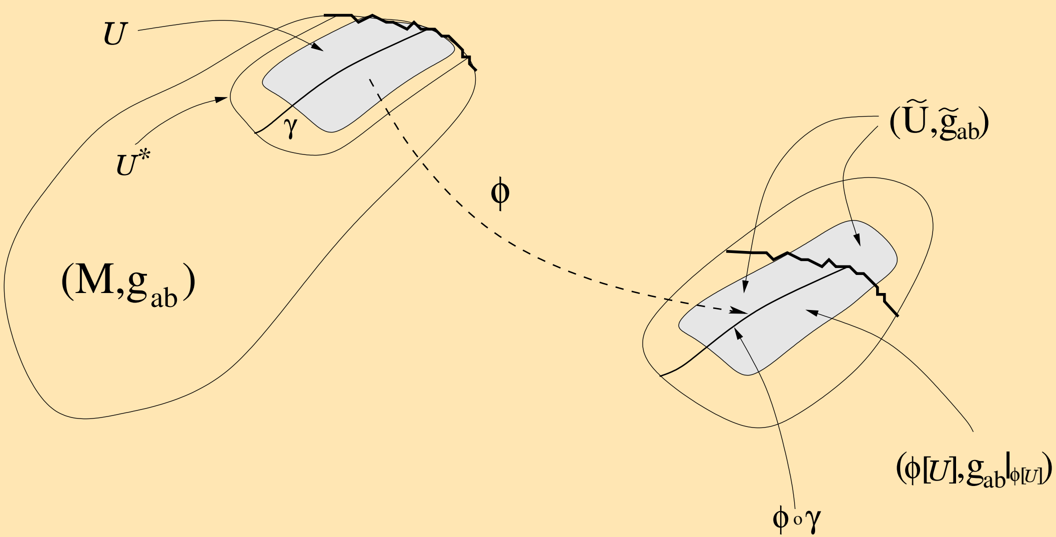
How the anticipated result would look like?

- **Theorem:** Consider a generic maximal globally hyperbolic timelike geodesically incomplete spacetime with a C^{1-} metric. Assume that the strong cosmic censor hypothesis holds for this class of spacetimes. Denote by γ one of the incomplete timelike geodesics. **Then, the tidal force components of the curvature tensor**—*measured with respect to parallelly propagated orthonormal frame fields, defined along a "synchronized" 3-parameter family of timelike geodesics ruling a neighbourhood of a final segment of* γ—**cannot be bounded.**

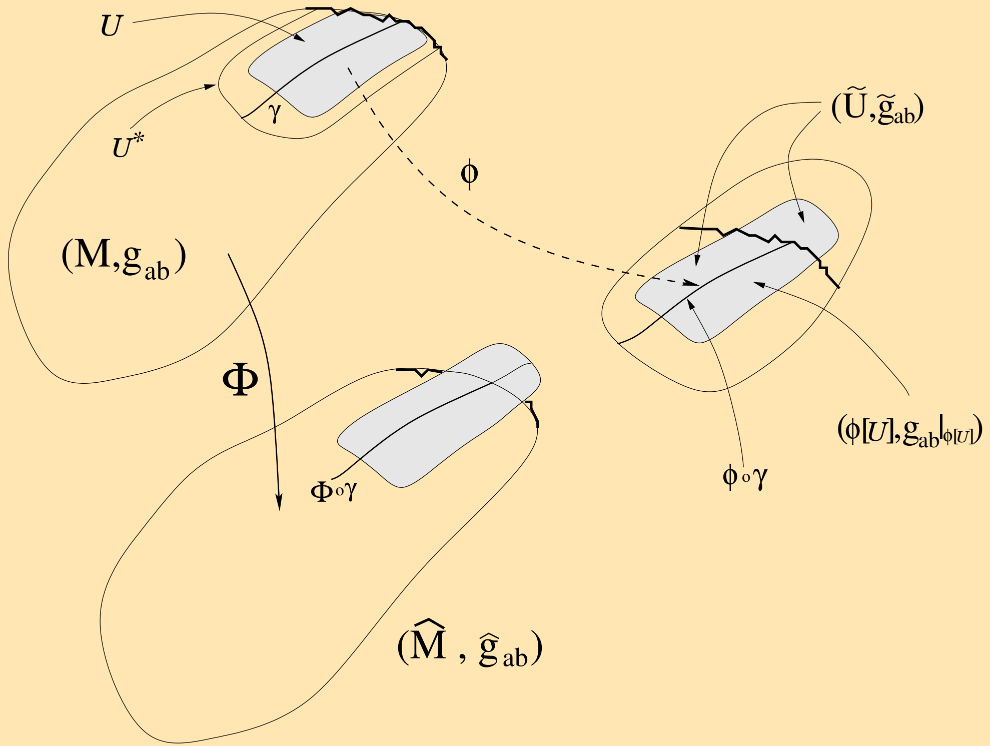
- would be interested in results relevant for the smooth setup the papers below could provide some useful hints
  - Rácz, I. (1993): *Spacetime extensions I.*, Journal of Mathematical Physics **34**, 2448-2464
  - Rácz, I. (2010): *Space-time extensions II*, Classical and Quantum Gravity **27**, 155007, Selected by the Editorial Board of Classical and Quantum Gravity (CQG) as part of the journal's Highlights of 2010, in 2011.

The 1st step in the construction

- start with a globally hyperbolic spacetime (M, g_{ab}), and
- assume that $\gamma : (t_1, t_*) \to M$ is a future incomplete timelike geodesic
  - let $\mathcal{U}^*$ be a neighbourhood of a final segment of γ
  - choose $\mathcal{U}$ to be a submanifold with boundary in $\mathcal{U}^*$
  - $\widetilde{\mathcal{U}}$: the union of $\phi[\mathcal{U}]$ and "a neighbourhood of the endpoint of $\phi \circ \gamma$"



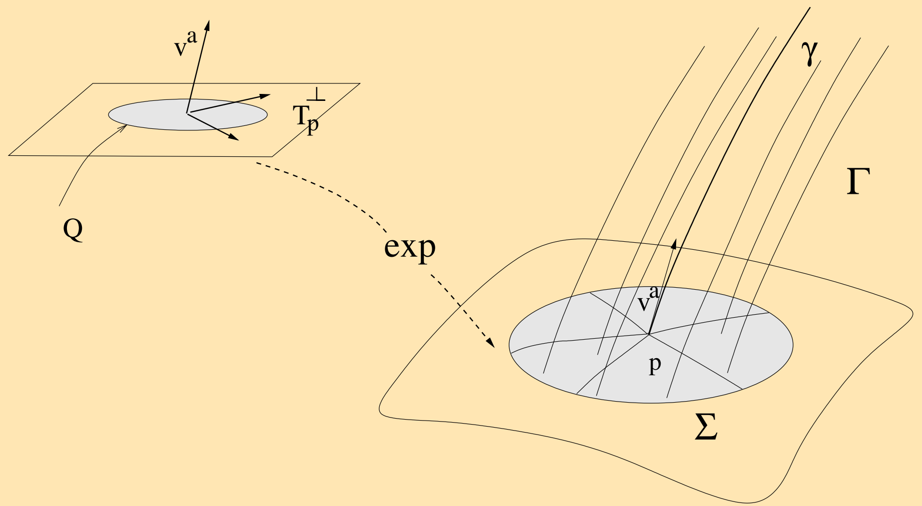
The 2nd step in the construction



- $(\widehat{M}, \widehat{g}_{ab})$: defined by gluing (M, g_{ab}) and $(\widetilde{\mathcal{U}}, \widetilde{g}_{ab})$ at their common parts
  - $\widehat{M}$ is the factor space $\widehat{M} = (M \cup \widetilde{\mathcal{U}})/\phi$, i.e it is yielded by the identification of points x, y in M and $\widetilde{\mathcal{U}}$ if and only if $x \in \mathcal{U}$ and $y \in \widetilde{\mathcal{U}}$ and $\phi(x) = y$
- $\widehat{g}_{ab}$ is the metric naturally induced by g_{ab} and $\widetilde{g}_{ab}$ on $\widehat{M}$

Construct $\mathcal{U}$ and then extend the subspacetime $(\mathcal{U}, g_{ab})$

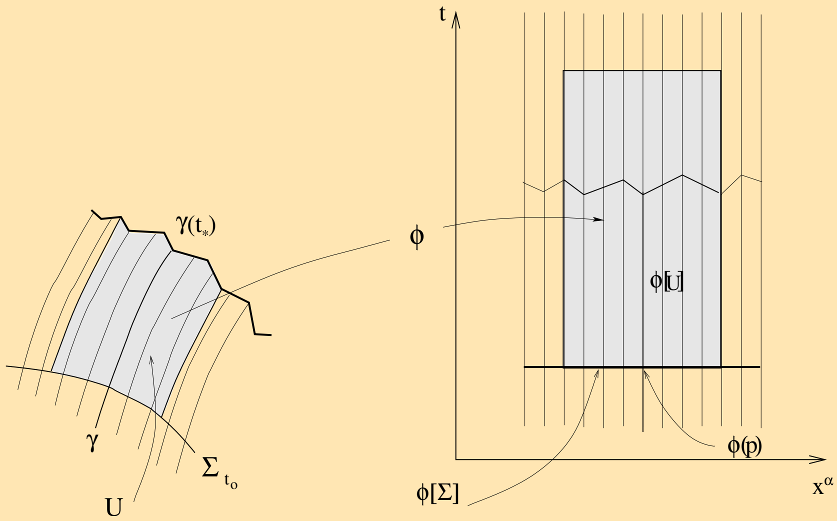
- the use of Whitney's theorem $\mathcal{U}$ requires a single coordinate patch
  - $\gamma : (t_1, t_2) \to M$ be a timelike geodesic & $p = \gamma(t_0)$; t is an affine parameter along γ, with tangent $v^a = (\partial/\partial t)^a$



  - let the hypersurface Σ be generated by spacelike geodesics starting at $p = \gamma(t_0)$ with tangent orthogonal to v^a
  - consider a "smooth" unit normal field v^a on Σ and the 3-parameter family of timelike geodesics, Γ, starting at Σ with tangent v^a
- Γ is a **"synchronized" 3-parameter family of timelike geodesics**
- $\mathcal{U}$ is ruled by members of Γ

How far away $\mathcal{U}$ extends from Σ?

- in a generic spacetime if the components of the Riemann tensor
$$R_{abcd}e^a_{(\mathfrak{a})}v^b e^c_{(\mathfrak{b})}v^d,$$
are bounded in **parallelly propagated orthonormal frames** $\{e^a_{(\mathfrak{a})}\}$, with $e^a_{(4)} = v^a$, along the synchronized 3-parameter congruence Γ
- then there exists $t_0 \in (t_1, t_*)$ such that $\mathcal{U}$ contains a final segment $\gamma|_{(t_0,t_*)}$ of γ, along with final segments of all the nearby members of Γ



- in addition, Gaussian coordinates $(x^1, x^2, x^3, x^4 = t)$ can be defined on $\mathcal{U}$, with metric $ds^2 = -dt^2 + g_{ij}\,dx^i dx^j$

We still need to extend the metric from $\phi[\mathcal{U}]$!!!

- Whitney's theorem $\Rightarrow$ the extendibility of functions defined on closed subsets of $\mathbb{R}^n$ $\Rightarrow$ the metric g_{ab} can be extended such that its extension $\widetilde{g}_{\alpha\beta}$ is of class C^{1-} if components g_{ij} can be shown to be Lipschitz functions on the closure $\overline{\phi[\mathcal{U}]}$ of $\phi[\mathcal{U}]$
- $\Rightarrow$ if the "t-derivatives" of $g_{ij} = g_{ab}\, E^a_{(i)} E^b_{(j)}$, where $E^a_{(i)} := (\partial/\partial x^i)^a$,
$$\partial_t g_{ij} = v^e \nabla_e [g_{ab}\, E^a_{(i)} E^b_{(j)}] = g_{ab} \left[\left(v^e \nabla_e E^a_{(i)} \right) E^b_{(j)} + E^a_{(i)} \left(v^e \nabla_e E^b_{(j)} \right) \right]$$
are uniformly bounded along the members of Γ
- it suffices to show that suitable norms of the coordinate basis fields $E^a_{(i)}$, and also that of $v^e \nabla_e E^a_{(i)}$ are uniformly bounded on $\phi[\mathcal{U}]$
- this, however, can be done by making use of the Jacobi equation
$$v^e \nabla_e \left(v^f \nabla_f E^a_{(\alpha)} \right) = R_{efg}{}^a\, v^e E^f_{(\alpha)} v^g$$
along with our indirect assumption on the boundedness of the tidal force components of the Riemann tensor, along the members of Γ
- Combining these $\Rightarrow$ in a generic globally hyperbolic timelike geodesically incomplete spacetime the tidal force components of the curvature tensor—measured by "synchronized" observes—cannot be bounded