Singularities in warped products: instability of extra dimensions?

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Outline

Introduction: Penrose on extra dimensions

In 2002, as a present to S.W. Hawking, Penrose argued that spatial compact extra-dimensions are likely to be unstable
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He ended up asserting:

> *(... a $4+n$-dimensional product spacetime) $M^4 \times \mathcal{Y}$ is highly unstable against small perturbations. If $\mathcal{Y}$ is compact and of Plack-scale size, then* spacetime singularities *are to be expected within a tiny fraction of a second!*

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To reach such conclusion he used the celebrated **singularity theorems**.

The classical Hawking-Penrose theorem

Theorem (Hawking and Penrose 1970)

If the convergence, causality and generic conditions hold and if there is one of the following:

- *a closed achronal set without edge,*
- *a closed trapped surface,*
- *a point with re-converging light cone*

then the space-time is causal geodesically incomplete.

Penrose's argument

- To use the singularity theorems, Penrose starts with a $(4+n)$-dimensional direct product $M_4 \times \mathcal{Y} = \mathbb{R} \times \mathbb{R}^3 \times \mathcal{Y}$ with metric as in e.g.

$$g = -dt^2 + dx^2 + dy^2 + dz^2 + g_{\mathcal{Y}}$$

and perturbs initial data given on a slice $\mathbb{R}^3 \times \mathcal{Y}$ (say $t = 0$) such that they do not 'leak out' into the $\mathbb{R}^3$-part: they only disturb the $\mathcal{Y}$-geometry.

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- He then forgets about the 3-dimensional typical space (in red) and considers a $(1+n)$-dimensional "reduced spacetime" $(\mathcal{Z}, g_{red})$ whose metric g_{red} is the evolution (e.g. Ricci-flat solution) of the initial data specified at $\mathcal{Y}$ $(t = 0)$.

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- the entire spacetime would be given by $\mathbb{R}^3 \times \mathcal{Z}$ with direct product metric

$$g_{pert} = g_{red} + dx^2 + dy^2 + dz^2$$

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- However, he claimed that such general disturbances are even more dangerous (due to the large approaching Planck-scale curvatures that are likely to be present in $\mathcal{Y}$).
- He defended that there is good reason to believe that these general perturbations will also result in spacetime singularities using again the H-P singularity theorem, but now using the *point with reconverging light cone* condition.

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There have been other arguments and discussions about the stability, or stabilization, of extra dimensions, but I want to concentrate here on those based on the existence of singularities.

The classical Hawking-Penrose theorem (again)

Theorem (Hawking and Penrose 1970)

If the convergence, causality and generic conditions hold and if there is one of the following:

- *a closed achronal set without edge, (co-dimension 1)*
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What about co-dimensions $3, \ldots, D-1$ — for instance, closed spacelike curves?

Trapped submanifolds of arbitrary dimension?

Some time ago, Galloway and I started to analyze the reasons behind the absence of other co-dimensions in the H-P singularity theorem, and we realized that the three conditions (on the point with reconverging light cone, on the closed trapped surface, and on the spacelike compact slice) could be unified into one single criterion of geometrical basis.

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The unification concept of trapping for arbitrary co-dimension:
$\Longrightarrow$ The mean curvature vector $\vec{H}$!

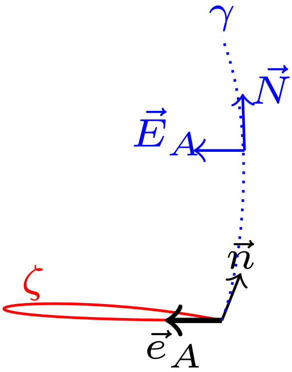
XXI-century singularity theorems

The parallel propagated projector $P^{\mu\nu}$

Notation

- n_μ: *future-pointing* normal to the spacelike submanifold ζ,
- γ: geodesic curve tangent to n^μ at ζ
- u: affine parameter along γ ($u = 0$ at ζ).
- N^μ: geodesic vector field tangent to γ ($N^\mu|_{u=0} = n^\mu$).
- $\vec{E}_A$: vector fields defined by parallel propagating $\vec{e}_A$ along γ ($\vec{E}_A|_{u=0} = \vec{e}_A$)
- By construction $g_{\mu\nu} E^\mu_A E^\nu_B$ is independent of u, so that $g_{\mu\nu} E^\mu_A E^\nu_B = g_{\mu\nu} e^\mu_A e^\nu_B = \gamma_{AB}$ (the 1st fundamental form of ζ).
- $P^{\nu\sigma} \equiv \gamma^{AB} E^\nu_A E^\sigma_B$ (at $u = 0$ this is the projector to ζ).

Notation on a picture



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Theorem (Generalized Hawking-Penrose singularity theorem)

If the chronology, generic and convergence conditions hold and there is a closed f-trapped submanifold ζ of arbitrary co-dimension such that

$$\boxed{R_{\mu\nu\rho\sigma}N^{\mu}N^{\rho}P^{\nu\sigma} \geq 0} \quad (1)$$

along every null geodesic emanating orthogonally from ζ then the spacetime is causal geodesically incomplete.

(G.J. Galloway and J.M.M. Senovilla, Singularity theorems based on trapped submanifolds of arbitrary co-dimension. *Class. Quantum Grav.* **27** (2010) 152002)

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- Spacelike 'surfaces' $m = 2$: Condition (1) is actually included in the convergence condition.
- Points $m = n$: The 'same' using the generic condition.

These three cases cover the original Hawking-Penrose theorem.

A generalized Penrose singularity theorem

Theorem

If $(\mathcal{V}, g)$ *contains a non-compact Cauchy hypersurface* Σ *and is null geodesically complete, then for every closed spacelike submanifold* ζ *there exists at least one null geodesic* γ *with initial tangent* n^μ *orthogonal to* ζ *along which*

$$\int_0^\infty R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} du \leq \theta(\vec{n}) \, .$$

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Observe that there is no restriction on the sign of $\theta(\vec{n})$.

Higher-dimensional spacetimes:

(warped) products

Direct product: "it just fails"

- Consider a spacetime $M = M_1 \times M_2$, $x^\mu = (x^a, x^i)$, with direct product metric

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- Let $\varsigma \subset M_2$ be compact and dim $\varsigma = k$, and let e^i_A be tangent vectors to ς. Then, their parallel transports along geodesics are such that $\vec{E}_A = (0, \overline{E}^i{}_A)$

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- Geodesics decompose too, tangent vectors are $\vec{N} = (\hat{N}^a, \overline{N}^i)$, with $\hat{N}^a$ and $\overline{N}^i$ geodesic in $(M_1, \hat{g})$ and $(M_2, \bar{g})$, respectively.

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- Then $R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} = \overline{R}_{ijkl} \overline{N}^i \overline{N}^k P^{jl}$, $P^{jl} = \gamma^{AB} E^j_A E^l_B$.

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- But, there are $\perp$ ς-null geodesics with $\overline{n}^i = \overline{N}^i(0) = 0$, and for these $\overline{N}^i(u) = 0$, and $\theta(\vec{n}) = 0$, so that any of the two conditions would read

$$0 > 0 \qquad \text{(just fails)}$$

Perturbations: warped products

- Consider perturbing the previous spacetime. The simplest way to do it (geometrically) is by breaking the direct product structure and letting one of the two pieces influence the other:

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- But they imply very different physical consequences!

Warped products: Curvature

- Recall $g_{\mu\nu}dx^\mu dx^\nu = \hat{g}_{ab}(x^c)dx^a dx^b + f^2(x^c)\bar{g}_{ij}(x^k)dx^i dx^j$

  $a, b, \ldots, h$ indices on M_1; i, j, k, l indices on M_2.

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- Recall $g_{\mu\nu}dx^\mu dx^\nu = \hat{g}_{ab}(x^c)dx^a dx^b + f^2(x^c)\bar{g}_{ij}(x^k)dx^i dx^j$

  $a, b, \ldots, h$ indices on M_1; i, j, k, l indices on M_2.

- $R^a{}_{ijk} = 0, \quad R^i{}_{abc} = 0, \quad R^i{}_{jab} = 0$
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- Observe that the unique solution with zero M_1-initial velocity $\hat{N}^a(0) = n^a = 0$ is $\hat{N}^a(u) = 0$.

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- In both cases, $C = 0$ means that the null geodesic lives exclusively in the Lorentzian part of the warped product.

Warped products: parallel transport

- As before, $\gamma : x^\mu = x^\mu(u)$ is an affinely parametrized geodesic (not necessarily null) with tangent vector $N^\mu = (\hat{N}^a, \bar{N}^i)$

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$$\bar{E}^i = f H \bar{N}^i + \frac{f_0}{f} \bar{E}^i_{\parallel},$$

$$\hat{N}^b \hat{\nabla}_b \hat{E}^a = -\left(\hat{g}_{bc} \hat{N}^b \hat{E}^c\right) \hat{\nabla}^a (\ln f)$$

where $H(u)$ is the unique solution of

$$\frac{dH}{du} = \hat{E}^a \partial_a (1/f), \qquad H(0) = 0$$

and $\bar{E}^i_{\parallel}$ is the parallel transport of e^i along the projected curve $\overline{\gamma} : x^i(u)$: $\bar{N}^j \overline{\nabla}_j \bar{E}^i_{\parallel} = 0, \qquad \bar{E}^i_{\parallel}(0) = e^i.$

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- One can first solve the eq. in red with $\hat{E}^a(0) = e^a$, then get $H(u)$ from the ODE in blue, and then get $\bar{E}^i$. **New**

Extra-dimension spreading: “just fails”

For the extra-dimension spreading over the Lorentzain part, either the latter is geodesically incomplete by itself or not, the extra dimensions being unable to turn it into null geodesically incomplete.

This follows from a known result that if the Riemannian base of a warped product is complete —which is always the case for compact base— then the spacetime is geodesically complete if and only if the fiber so is.

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Thus, Penrose's suggestion that "disturbances that significantly spill over into the 4-dimensional part of the spacetime" would be more dangerous and will result in singularities does not seem to sustain —at least in this warped-product situation.

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- $g_{\mu\nu}E^{\mu}_B E^{\nu}_A = \delta_{BA} \quad \Longrightarrow \quad \bar{g}_{ij}\bar{E}^i_A\bar{E}^j_B = (1/f^2)\delta_{AB}$.

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- In this case the tensor $P^{\mu\nu} = \gamma^{AB} E^\mu_A E^\nu_B$ reads

$$P^{ab} = 0, \quad P^{ia} = 0, \quad P^{ij} = \frac{1}{f^2} \delta^{AB} \bar{E}^i_{A\|} \bar{E}^j_{B\|}$$

Expression (1), case ❷

- $R_{\mu\nu\rho\sigma}N^{\mu}N^{\rho}P^{\nu\sigma} = \delta^{AB}\bar{R}_{ijkl}\bar{N}^{i}\bar{N}^{k}\bar{E}^{j}_{A\|}\bar{E}^{l}_{B\|} - (D-m)\frac{1}{f}\frac{d^2 f}{du^2}|_{\gamma}$

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- A simple computation gives, for the expansion along $\vec{n}$:

$$\theta(\vec{n}) = \bar{\theta}_{\bar{n}} + (D-m) \frac{1}{f_0} \frac{df}{du}(0)$$

where $\bar{\theta}_{\bar{n}}$ is "expansion of ζ as submanifold of $(M_2, \bar{g})$".

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- The integrated condition in the singularity theorem reads then

$$\int_0^\infty \left(\delta^{AB}\overline{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j_{A\|}\bar{E}^l_{B\|} - (D-m)\frac{1}{f}\frac{d^2 f}{du^2}|_\gamma\right) du \leq \bar{\theta}_{\bar{n}} + (D-m)\frac{1}{f_0}\frac{df}{du}(0)$$

Singularity theorems in warped products

Theorem

Let $M = M_1 \times_f M_2$ be a null geodesically complete D-dimensional warped product spacetime with metric

$$g_{\mu\nu}dx^\mu dx^\nu = \hat{g}_{ab}(x^c)dx^a dx^b + f^2(x^c)\bar{g}_{ij}(x^k)dx^i dx^j$$

containing a non-compact Cauchy hypersurface. Then, every compact submanifold $\zeta \subset M_2$, of any possible co-dimension m, launches at least one future-directed null geodesic emanating orthogonally to ζ satisfying the inequality

$$\int_0^\infty \left(\delta^{AB}\overline{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j_{A\|}\bar{E}^l_{B\|} - (D-m)\frac{1}{f}\frac{d^2 f}{du^2}|_\gamma\right) du \leq \bar{\theta}_{\bar{n}} + (D-m)\frac{1}{f_0}\frac{df}{du}(0)\,.$$

Analysis of the inequality condition

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- The violation of this condition is, therefore, necessary (along all null geodesics with $C = 0$) for the singularities to appear.

dynamical perturbations: instability?

- If the extra dimensions start, say, contracting —otherwise, use past version— along M_1-null directions (i.e. $\hat{N}^a \hat{\nabla}_a f(0) \leq 0$) then it is enough that the Hessian of f be non-positive on those null directions *on average*.

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- Observe that, from the expression of the Ricci tensor and as $N^\mu = (\hat{N}^a, 0)$ for these null geodesics, the null energy condition (NEC) on them reads

$$R_{\mu\nu} N^\mu N^\nu = \hat{R}_{ab} \hat{N}^a \hat{N}^b - n \frac{1}{f} \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f \geq 0$$

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- This immediately implies

$$-\int_\gamma \frac{1}{f}\hat{N}^a\hat{N}^b\hat{\nabla}_a\hat{\nabla}_b f \geq \int_\gamma \hat{R}_{ab}\hat{N}^a\hat{N}^b \geq 0$$

which is always non-negative if the NEC holds on average in the noticeable, observed, 4-dimensional spacetime.

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- The last summand here tends to be non-positive, due to the above and because $\hat{N}^a$ are now timelike

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- $d^2 f/du^2|_\gamma = (C/f^3)\hat{\nabla}^b f \hat{\nabla}_b f + \hat{N}^a \hat{N}^b \hat{\nabla}_a \hat{\nabla}_b f$
- The last summand here tends to be non-positive, due to the above and because $\hat{N}^a$ are now timelike
- The first summand on the righthand side favors the singularity if the gradient of f is non-spacelike: this is the case if the perturbation is truly dynamical (i.e., the dynamical part dominates over other possible accompanying perturbations).

dynamical perturbations: instability!?

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- The first summand on the righthand side favors the singularity if the gradient of f is non-spacelike: this is the case if the perturbation is truly dynamical (i.e., the dynamical part dominates over other possible accompanying perturbations).
- Actually, keeping the values of the coupling constants (and the Planck mass) independent of position in space implies f should depend only on time and thus $\hat{\nabla}^b f \hat{\nabla}_b f < 0$.

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- $\bar{\theta}_{\bar{n}}$ is a quantity relative to the extra-dimension space too.

Concluding remarks

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- If the co-dimension is 5, that is dimension $n-1$, then $\delta^{AB}\bar{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j_{A\|}\bar{E}^l_{B\|} = \bar{R}_{ij}\bar{N}^i\bar{N}^j$ (Ricci-flat M_2 OK!)

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- Or if dim $\zeta = 1$, i.e. a circle, then $\delta^{AB}\bar{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j_{A\|}\bar{E}^l_{B\|} = \bar{R}_{ijkl}\bar{N}^i\bar{N}^k\bar{E}^j\bar{E}^l$ is just a sectional curvature.

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- As M_2 is compact, the integral may be the sum of an infinite number of integrals on closed geodesics.

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- In essence, *dynamical* perturbations will generically lead to the appearance of singularities, destroying the stationary classical stability of the extra-dimensional space.
- On a positive side, the condition as given involving quantities of only the extra-dimensional space may help in finding the stable possibilities, providing information on which classes of compact extra-dimensions —and for which warping functions $f(t)$— are viable and why.

Thanks

Thank you for your attention!

dziękuję !