What happens to spikes after their quantization?

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Outline:

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Context and motivation

- BKL conjecture means decoupling of points close to a singularity
- Sharp features (called spikes) develop at some individual points
- In cosmology, spikes are potential seeds for matter inhomogeities
- It is still unclear whether they survive after the quantization

Our gravitational variables are the spatial connection C_i^j and extrinsic curvature D_i^j, both contracted with the densitized triad. The independent DOFs in the BKL regime are their eigenvalues C_I, D_I, where $I = 1, 2, 3$. C_I and D_I form the affine Poisson (Lie) algebra

$$\{C_I, C_J\} = 0 = \{D^I, D^J\}, \qquad \{C_I, D^J\} = \delta_I^J C_J, \tag{1}$$

as introduced in A. Ashtekar, A. Henderson and D. Sloan, Phys. Rev. D **83**, 084024 (2011).

Dynamics parametrized by a scalar field

The Hamiltonian (3) of the system of (C_I, D_I) and the uncoupled scalar field (ϕ, π) (acting as a clock) determines the equations of motion

$$\dot{D}_I = -C_I\Big(\sum_J C_J - 2C_I\Big)\,, \qquad \dot{C}_I = 4C_I\Big(\sum_J D_J - 2D_I\Big)\,,$$
$$\dot{\pi} = 0\,, \qquad \dot{\phi} = \kappa\pi\,, \tag{2}$$

where the sign $\kappa = \pm 1$. We reparametrize time via the transformation $\frac{d}{dt} = \kappa\pi\frac{d}{d\phi}$. Moreover, there is the Hamiltonian constraint

$$\frac{1}{2}\Big(\sum_I C_I\Big)^2 - \sum_I C_I^2 + 4\left(\frac{1}{2}\Big(\sum_I D_I\Big)^2 - \sum_I D_I^2\right) + \frac{\kappa}{2}\pi^2 = 0\,. \tag{3}$$

If we assume $0 < C_I \ll 1$ and $D_1 < D_2 < D_3 < 0$ at the initial "time" $\phi = \phi_0$, then it turns out that C_2 and C_3 quickly vanish, while D_2, D_3 are essentially constant. Consequently, the whole system reduces to

$$\kappa\pi C_1' = 4C_1(-D_1 + D_2 + D_3)\,, \qquad \kappa\pi D_1' = C_1^2\,,$$
$$C_1^2 = -4(D_1 - D_+)(D_1 - D_-) + \kappa\pi^2\,, \tag{4}$$

Solution of the gravitational dynamics

where $D_{\pm} \equiv D_2 + D_3 \pm 2\sqrt{D_2 D_3}$. We find that solutions for C_1 and D_1 have the form (introducing the notation $\kappa\pi^2 + 16 D_2 D_3 \equiv g(D_2, D_3; \pi)$)

$$C_1(\phi) = \mathrm{sgn}(C_{10})\sqrt{g(D_2, D_3; \pi)}\,\mathrm{sech}\left[\frac{2}{\kappa\pi}\sqrt{g(D_2, D_3; \pi)}\,(\phi - \phi_0) - \mathrm{arctanh}\sqrt{\frac{g(D_2, D_3; \pi) - C_{10}^2}{g(D_2, D_3; \pi)}}\right],$$

$$D_1(\phi) = D_2 + D_3 + \frac{1}{2}\sqrt{g(D_2, D_3; \pi)}\,\tanh\left[\frac{2}{\kappa\pi}\sqrt{g(D_2, D_3; \pi)}\,(\phi - \phi_0) - \mathrm{arctanh}\sqrt{\frac{g(D_2, D_3; \pi) - C_{10}^2}{g(D_2, D_3; \pi)}}\right], \tag{5}$$

where the initial condition $C_1(\phi_0) = C_{10}$ appears (and $C_1(\phi) = 0$ for $C_{10} = 0$), while $D_{10} = D_1(\phi_0)$ has been eliminated by the constraint.

Emergence of a spike

Let us set a spatial coordinate $x = 0$ at the point where $C_{10} = 0$. Choosing a simple parametrization $C_{10} := x$, as well as $\kappa = 1$, $\phi_0 = 0$, we may now draw $C_1(x)$ and $D_1(x)$.

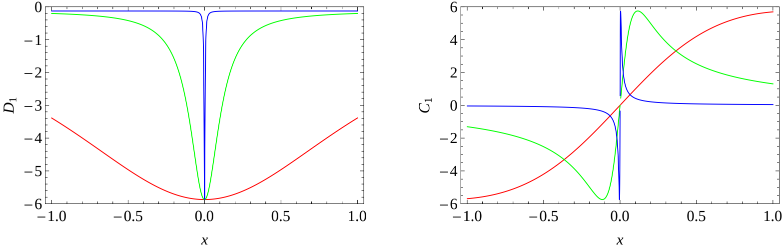

Figure: $D_1(x)$ and $C_1(x)$ for evolution parameters $\phi = 0.2$ (red), $\phi = 0.4$ (green) and $\phi = 0.7$ (blue)

Characterizing the spike by a single curve

The arc length of the curve $\vec{r}(x) \equiv (C_1(x), D_1(x))$ is given by

$$
\begin{aligned}
s(x) &= \int_{x_0}^{x} d\tilde{x} \sqrt{\left(\frac{dC_1(\tilde{x})}{d\tilde{x}}\right)^2 + \left(\frac{dD_1(\tilde{x})}{d\tilde{x}}\right)^2} \\
&= -\frac{1}{2}\sqrt{g(D_2, D_3; \pi)} \left(i\mathrm{E}(iz, 4) - i\mathrm{F}(iz, 4) + \frac{\sqrt{2\cosh(2z) - 1}}{\coth z} \right)_{x_0}^{x}, \\
z &\equiv \frac{2}{\kappa\pi}\sqrt{g(D_2, D_3; \pi)}\,(\phi - \phi_0) - \mathrm{arctanh}\frac{\tilde{x}}{\sqrt{g(D_2, D_3; \pi)}}\,.
\end{aligned} \tag{6}
$$

In terms of the so-called Frenet vectors $\hat{e}_1(s)$, $\hat{e}_2(s)$ (whose form is derived from $\vec{r}(s)$), the curvature of $\vec{r}(s)$ is defined as

$$
\chi(s) := \left|\frac{\vec{r}(s)}{ds}\right|^{-1} \frac{d\hat{e}_1(s)}{ds} \cdot \hat{e}_2(s)\,. \tag{7}
$$

Emergence of a spike – another view

The normalized arc length $\bar{s}$ corresponding to $x \in [x_0, x_1]$ for a given ϕ is obtained by dividing $s(x)$ by its maximal value $s(x_1)$.

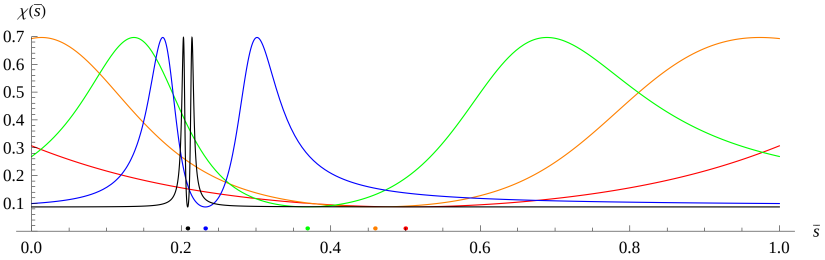

Figure: Curvature of the curve $(C_1(\bar{s}), D_1(\bar{s}))$ for evolution parameters $\phi = 0$ (red), $\phi = 0.05$ (orange), $\phi = 0.1$ (green), $\phi = 0.2$ (blue), and $\phi = 0.4$ (black)

Dots on the horizontal axis denote the value of $\bar{s}(x = 0)$ for a given ϕ.

Quantization of the gravitational DOFs

The direct quantization of the affine Poisson algebra of (C_I, D_I) gives

$$[\widehat{C}_I, \widehat{C}_J] = 0 = [\widehat{D}^I, \widehat{D}^J]\,, \qquad [\widehat{C}_I, \widehat{D}^J] = i\,\delta_I^J \widehat{C}_J\,. \tag{8}$$

Each pair of operators $\widehat{C}_I$, $\widehat{D}^I$ can be represented on the Hilbert space $\mathcal{H}^I := L^2(\mathbb{R}_-, d\nu(x^I)) \oplus L^2(\mathbb{R}_+, d\nu(x^I))$ (with $d\nu(x^I) := dx^I/|x^I|$) as

$$\widehat{C}_I \psi(x^I) := x^I \psi(x^I)\,, \qquad \widehat{D}^I \psi(x^I) := -ix^I \frac{\partial}{\partial x^I}\, \psi(x^I)\,. \tag{9}$$

Furthermore, the affine algebra generates the corresponding group $\mathrm{Aff}(\mathbb{R})$. Elements of its unitary representation on $\mathcal{H}^I$ are

$$\widehat{U}(p^I, q_I) = e^{ip^I \widehat{C}_I} e^{i \ln |q_I| \widehat{D}^I}\,, \tag{10}$$

where $p^I \in (-\infty, +\infty)$, $|q_I| \in (0, +\infty)$. The action of $\widehat{U}(p^I, q_I)$ on $\psi \in \mathcal{H}^I$ turns out to have the "irreducible" form

$$\widehat{U}(p^I, q_I)\, \psi(x^I) = \Theta(q_I x^I) e^{ip^I x^I} \psi(|q_I| x^I)\,. \tag{11}$$

Introducing affine coherent states

We consider integrals over $\mathrm{Aff}(\mathbb{R})$ with the left-invariant measure,

$$\int_{\mathrm{Aff}(\mathbb{R})} d\mu_L(p^I, q_I)\,(\ldots) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} dp^I \int_0^{+\infty} \frac{dq_I}{q_I^2}\,(\ldots)\,. \qquad (12)$$

From a given fiducial vector $\Phi_\pm^{(I)} \in L^2(\mathbb{R}_\pm, d\nu(x^I))$, we may now construct a family of coherent states $\widehat{U}(p^I, q_I)\,|\Phi_\pm^{(I)}\rangle$. Then there exists a non-orthogonal decomposition of the identity operator

$$\mathbb{I}_\pm^{(I)} = \frac{1}{A_{\Phi_\pm^{(I)}}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu_L(p^I, q_I)\,\widehat{U}(p^I, q_I)|\Phi_\pm^{(I)}\rangle\langle\Phi_\pm^{(I)}|\widehat{U}(p^I, q_I)^\dagger\,, \qquad (13)$$

where $A_{\Phi_\pm^{(I)}} \equiv \int_0^\infty dx^I/(x^I)^2|\Phi_\pm^{(I)}(x^I)|^2$. It allows to decompose the identity $\mathbb{I}_{123} = \bigotimes_I(\mathbb{I}_-^{(I)} + \mathbb{I}_+^{(I)})$ of the total Hilbert space $\mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \mathcal{H}^3$.

Constructing quantum operators

The unitary operators are simply extended to $\mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \mathcal{H}^3$, so that

$$\bigotimes_I \widehat{U}(p^I, q_I)\,\psi(x^1, x^2, x^3) \tag{14}$$

$$= \prod_I \Theta(q_I x^I) e^{i \sum_I p^I x^I} \psi(|q_1| x^1, |q_2| x^2, |q_3| x^3)\,.$$

Meanwhile, using a decomposition of $\mathbb{I}_{123}$, a function f on $\mathrm{Aff}(\mathbb{R})^{\otimes 3}$ can be mapped to the quantum observable

$$\widehat{f} := \int_{\mathrm{Aff}(\mathbb{R})^{\otimes 3}} \prod_I d\mu_L(p^I, q_I)\, f(p_1, p_2, p_3, q_1, q_2, q_3)$$

$$\bigotimes_{I=1}^{3} \sum_{\sigma \in \{\pm\}} \frac{1}{A_{\Phi_\sigma^{(I)}}} \widehat{U}(p^I, q^I) |\Phi_\sigma^{(I)}\rangle \langle \Phi_\sigma^{(I)}| \widehat{U}(p^I, q^I)^\dagger\,, \tag{15}$$

which is self-adjoint if (but not only) $f \in L^1(\mathrm{Aff}(\mathbb{R})^{\otimes 3}, \prod_I d\mu_L(p^I, q_I))$.

Classical clock for a quantum system

The adjoint action (given by the Poisson bracket) of π on any function f on the field's phase space generates a shift in ϕ via

$$e^{\tau\{\cdot,\pi\}}f(\phi,\pi) \equiv e^{\tau\frac{\partial}{\partial\phi}}f(\phi,\pi) = f(\phi+\tau,\pi)\,. \tag{16}$$

Similarly, for a wave function of our gravitational DOFs, the "time" parameter ϕ can be formally shifted via

$$e^{\tau\tilde{\pi}}\Psi(x;\phi) := e^{\tau\frac{\partial}{\partial\phi}}\Psi(x;\phi) = \Psi(x;\phi+\tau)\,. \tag{17}$$

On the other hand, such a shift should be performed by the evolution operator, $\widehat{U}(\tau)\Psi(x;\phi) = \Psi(x;\phi+\tau)$. We postulate that

$$\widehat{U}(\tau) = \widehat{U}_{\mathcal{K}}(\tau)V_{\tilde{\pi}}(\tau) := \widehat{U}_{\mathcal{K}}(\tau)e^{-i\tau E(\tilde{\pi})}\,, \tag{18}$$

where $\widehat{U}_{\mathcal{K}}(\tau)$ is a unitary operator acting on $\mathcal{K} \ni \Psi(x;\phi)$ and $E(\tilde{\pi})$ is a (classical) real function of $\tilde{\pi} = \frac{\partial}{\partial\phi}$.

Schrödinger-like equation for the system

Comparing both expressions for $\Psi(x;\phi+\tau)$ and taking the derivative of their equality at $\tau=0$, one obtains

$$i\frac{\partial}{\partial\phi}\Psi(x;\phi)=\left(\widehat{W}+E\left(\frac{\partial}{\partial\phi}\right)\widehat{\mathbb{I}}\right)\Psi(x;\phi)\,, \tag{19}$$

where $\widehat{W}:=i\left(\frac{\partial\widehat{U}_{\mathcal{K}}(\tau)}{\partial\tau}\right)_{\tau=0}$. Since the Hamiltonian should amount to $\widehat{H}=\widehat{W}+E(\frac{\partial}{\partial\phi})\widehat{\mathbb{I}}$ (and there are no ordering issues) we find that

$$\widehat{W}=2\Big(\sum_I x_I^2\frac{\partial^2}{\partial x_I^2}-2\sum_{I<J}x_Ix_J\frac{\partial^2}{\partial x_I\partial x_J}+\sum_I x_I\frac{\partial}{\partial x_I}\Big)$$
$$+\sum_{I<J}x_Ix_J-\frac{1}{2}\sum_I x_I^2\,,\qquad E\left(\frac{\partial}{\partial\phi}\right)=-\frac{\kappa}{2}\frac{\partial^2}{\partial\phi^2}\,. \tag{20}$$

Finally, assuming the separability of $\Psi(x;\phi)=\omega(\phi)\psi(x)$, one can derive a pair of eigenequations

$$\Big(i\frac{\partial}{\partial\phi}+\frac{\kappa}{2}\frac{\partial^2}{\partial\phi^2}\Big)\omega_\lambda(\phi)=\lambda\,\omega_\lambda(\phi)\,,\qquad \widehat{W}\psi_\lambda(x)=\lambda\,\psi_\lambda(x)\,. \tag{21}$$

Work in progress and outlook

Solutions of the first eigenequation are found to be

$$\omega_\lambda(\phi) = e^{-i\kappa\phi}\left(A_\lambda e^{\kappa\sqrt{2\kappa\lambda-1}\,\phi} + B_\lambda e^{-\kappa\sqrt{2\kappa\lambda-1}\,\phi}\right) \tag{22}$$

if $\lambda \neq \kappa/2$, whereas for $\lambda = \kappa/2$ we have

$$\omega_\lambda(\phi) = (A_\lambda \phi + B_\lambda) e^{-i\kappa\phi}\,. \tag{23}$$

The form of these solutions means that $\Psi(x;\phi) = 0$ unless the constraint $\widehat{H}\,\Psi(x;\phi) = 0$ is replaced by

$$\left\langle \widehat{W} + E\left(\frac{\partial}{\partial\phi}\right)\right\rangle_\Psi = 0\,. \tag{24}$$

The other eigenequation seems to require numerical methods. The results of our analysis shall be compared with E. Czuchry, D. Garfinkle, J. R. Klauder and W. Piechocki, Phys. Rev. D **95**, 024014 (2017).